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Bregman strongly nonexpansive operators in reflexive Banach spaces

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ABSTRACT

We present a detailed study of right and left Bregman strongly nonexpansive operators in reflexive Banach spaces. We analyze, in particular, compositions and convex combinations of such operators, and prove the convergence of the Picard iterative method for operators of these types. Finally, we use our results to approximate common zeros of maximal monotone mappings and solutions to convex feasibility problems.

1. Introduction

The theory and applications of nonexpansive operators in Banach spaces have been intensively studied for almost fifty years now [6,22–24]. There are several important classes of nonexpansive operators which have remarkable properties not shared by all such operators. We refer, for example, to strongly nonexpansive operators which were introduced in [15]. This class of operators is of particular significance in fixed point, iteration and convex optimization theories mainly because it is closed under composition. It encompasses other noteworthy classes of nonexpansive operators. For example, in uniformly convex Banach spaces all firmly nonexpansive operators as well as all averaged operators are strongly nonexpansive [15]. A related class of operators comprises the quasi-nonexpansive operators. These operators still enjoy relevant fixed point properties although nonexpansivity is only required for each fixed point [21].

In this paper, we are concerned with certain analogous classes of operators which are, in some sense, strongly nonexpansive not with respect to the norm, but with respect to Bregman distances [2,14,17,20]. Since these distances are not symmetric in general, it seems natural to distinguish between left and right Bregman strongly nonexpansive operators. The left variant has already been studied and applied in [30,32]. We have recently introduced and studied several classes of right Bregman nonexpansive operators in reflexive Banach spaces [26,27]. The present paper is devoted to a detailed study of right and left Bregman strongly nonexpansive operators.

Our paper is organized as follows. Section 2 contains certain essential preliminary results regarding properties of Bregman distances. In the next section we establish in detail two fundamental properties of left Bregman strongly nonexpansive operators. These properties concern the compositions of finitely many such operators. In Section 4 we establish analogous

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results for right Bregman strongly nonexpansive operators. The next section is devoted to convex combinations of a given finite number of right Bregman strongly nonexpansive operators. In Section 6 we bring out the connections between left and right Bregman strongly nonexpansive operators. Finally, in the last two sections we prove the convergence of the Picard iteration for right Bregman strongly nonexpansive operators and then use our results to find common zeros of maximal monotone mappings and to solve convex feasibility problems.

2. Preliminaries

In this section we collect several definitions and results, which are pertinent to our study. Let X be a reflexive Banach space and let $f : X \to (-\infty, +\infty]$ be a function with effective domain $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$. The Fenchel conjugate function $f^* : X^* \to (-\infty, +\infty]$ of f is defined by $f^*(u) = \sup\{\langle u, x\rangle - f(x) : x \in \operatorname{dom} f\}$. We say that f is *admissible* if it is proper, convex, lower semicontinuous and Gâteaux differentiable on $\operatorname{int} \operatorname{dom} f$. Under these assumptions we know that f is continuous in $\operatorname{int} \operatorname{dom} f$ (see [4, Fact 2.3, p. 619]). Recall that f is called *cofinite* if $\operatorname{dom} f^* = X^*$. The subdifferential of f is the set-valued mapping $\partial f : X \to 2^{X^*}$ defined by

$$\partial f(x) := \left\{u \in X^* : f(y) \geq f(x) + \langle u, y - x\rangle \ \forall y \in X\right\}, \quad x \in X.$$

The boundedness of f is inherited by the subdifferential and *vice versa*, as the following result shows.

Proposition 2.1 (*Cf. [17, Proposition 1.1.11, p. 16]*)**.** *If $f : X \to \mathbb{R}$ is continuous and convex, then $\partial f : X \to 2^{X^*}$ is bounded on bounded subsets if and only if f itself is bounded on bounded subsets.*

The function f is said to be *Legendre* if it satisfies the following two conditions.

(L1) $\operatorname{int} \operatorname{dom} f \neq \emptyset$ and ∂f is single-valued on its domain ($\operatorname{dom} \partial f = \{x \in X : \partial f(x) \neq \emptyset\}$).
(L2) $\operatorname{int} \operatorname{dom} f^* \neq \emptyset$ and ∂f^* is single-valued on its domain.

When the subdifferential of f is single-valued, it coincides with the gradient $\partial f = \nabla f$ [29, Definition 1.3, p. 3] (this is the case when f is an admissible function, not necessarily Legendre). The class of Legendre functions in infinite dimensional Banach spaces was first introduced and studied by Bauschke, Borwein and Combettes in [4]. Their definition is equivalent to conditions (L1) and (L2) because the space X is assumed to be reflexive (see [4, Theorems 5.4 and 5.6, p. 634]). It is well known that in reflexive spaces $\nabla f = (\nabla f^*)^{-1}$ (see [4, Theorem 5.10, p. 636]). When this fact is combined with conditions (L1) and (L2), we obtain

$$\operatorname{ran} \nabla f = \operatorname{dom} \nabla f^* = \operatorname{int} \operatorname{dom} f^* \quad \text{and} \quad \operatorname{ran} \nabla f^* = \operatorname{dom} \nabla f = \operatorname{int} \operatorname{dom} f.$$

It also follows that f is Legendre if and only if f^* is Legendre (see [4, Corollary 5.5, p. 634]) and that the functions f and f^* are Gâteaux differentiable and strictly convex in the interior of their respective domains. When the Banach space X is smooth and strictly convex, in particular, a Hilbert space, the function $(1/p)\|\cdot\|^p$ with $p \in (1, \infty)$ is Legendre (*cf.* [4, Lemma 6.2, p. 639]). For examples and more information regarding Legendre functions, see, for instance, [3,4].

The bifunction $D_f : \operatorname{dom} f \times \operatorname{int} \operatorname{dom} f \to [0, +\infty)$ given by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle \tag{1}$$

is called the *Bregman distance with respect to* f (*cf.* [19]). With the function f we associate the bifunction $W^f : \operatorname{dom} f^* \times \operatorname{dom} f \to [0, +\infty)$ defined by

$$W^f(\xi, x) := f(x) - \langle \xi, x\rangle + f^*(\xi). \tag{2}$$

This function satisfies

$$W^f(\nabla f(x), y) = D_f(y, x)$$

for all $x \in \operatorname{int} \operatorname{dom} f$ and $y \in \operatorname{dom} f$ (*cf.* [28]).

We now recall the definition of a totally convex function which was introduced in [16,17].

Definition 2.2 (*Total Convexity*)**.** The function f is called *totally convex at a point* $x \in \operatorname{int} \operatorname{dom} f$ if its *modulus of total convexity at* x, $\nu_f(x, \cdot) : [0, +\infty) \to [0, +\infty]$, defined by

$$\nu_f(x, t) := \inf\left\{D_f(y, x) : y \in \operatorname{dom} f,\ \|y - x\| = t\right\},$$

is positive whenever $t > 0$. The function f is called *totally convex* when it is totally convex at every point of $\operatorname{int} \operatorname{dom} f$.

Definition 2.3 (*Total Convexity on Bounded Subsets*)**.** The function f is called *totally convex on bounded sets* if, for any nonempty and bounded set $E \subset X$, the *modulus of total convexity of* f on E, $\nu_f(E, t)$, is positive for any $t > 0$, where $\nu_f(E, \cdot) : [0, +\infty) \to [0, +\infty]$ is defined by

$$\nu_f(E, t) := \inf\left\{\nu_f(x, t) : x \in E \cap \operatorname{int} \operatorname{dom} f\right\}.$$

Relevant examples of functions f with the above properties can be found in [5,10–13,18,26,35].

The following proposition will play a crucial role in our results (*cf.* [17, Lemma 2.1.2, p. 67]).