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Precompact groups and property (T)

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ABSTRACT

For a topological group G, the dual object $\widehat{G}$ is defined as the set of equivalence classes of irreducible unitary representations of G equipped with the Fell topology. It is well known that, if G is compact, $\widehat{G}$ is discrete. In this paper, we investigate to what extent this remains true for precompact groups, that is, dense subgroups of compact groups. We show that: (a) if G is a metrizable precompact group, then $\widehat{G}$ is discrete; (b) if G is a countable non-metrizable precompact group, then $\widehat{G}$ is not discrete; (c) every non-metrizable compact group contains a dense subgroup G for which $\widehat{G}$ is not discrete. This extends to the non-Abelian case what was known for Abelian groups. We also prove that, if G is a countable Abelian precompact group, then G does not have Kazhdan's property (T), although $\widehat{G}$ is discrete if G is metrizable.

1. Introduction

For a topological group G, let $\widehat{G}$ be the set of equivalence classes of irreducible unitary representations of G. The set $\widehat{G}$ is equipped with the so-called Fell topology [11], which can be defined on every set of equivalence classes of not necessarily irreducible representations of a given topological group G. We recall some familiar cases (see [23]):

(1) if G is Abelian, then $\widehat{G}$ is the standard Pontryagin–van Kampen dual group, and the Fell topology on $\widehat{G}$ is the usual compact-open topology;
(2) when G is compact, the Fell topology on $\widehat{G}$ is the discrete topology;
(3) when $\widehat{G}$ is neither Abelian nor compact, $\widehat{G}$ usually is non-Hausdorff.

In general, little is known about the properties of the Fell topology. Its understanding heavily depends on harmonic analysis. For example, if G is a second countable locally compact group, then the Fell topology on $\widehat{G}$ satisfies the T_1 separation axiom if and only if G is type I (see [3]).

It is possible to define a different topology on $\widehat{G}$ that, when G is compact, is *grosso modo* the natural quotient of the set of all irreducible representations equipped with the compact-open topology. However, this topology is less useful than the Fell topology, because for every integer n it makes the set of all n-dimensional representations closed, while it is often desirable to regard certain lower-dimensional representations as limits of higher-dimensional ones (see [3,9,7,22,23]).

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A topological group G is *precompact* if it is isomorphic (as a topological group) to a subgroup of a compact group H. If G is compact, then $\widehat{G}$ is discrete. If G is a dense subgroup of H, the natural mapping $\widehat{H} \to \widehat{G}$ is a bijection but in general need not be a homeomorphism. Following Comfort, Raczkowski, and Trigos-Arrieta [6], we say that G *determines* H if the natural bijection $\widehat{H} \to \widehat{G}$ is a homeomorphism, that is, $\widehat{G}$ is discrete. A compact group H is *determined* if every dense subgroup of H determines H. In this paper, we investigate the following question: if H is a compact group and G is a dense subgroup of H, under what conditions does G determine H? Equivalently, for what precompact groups G is $\widehat{G}$ discrete?

In the Abelian case, this question has been settled in the work of several authors. Aussenhofer [2] and, independently, Chasco [4] showed that every metrizable Abelian compact group H is determined. Comfort, Raczkowski, and Trigos-Arrieta [6] noted that the Aussenhofer–Chasco theorem fails for non-metrizable Abelian compact groups H. More precisely, they proved that every non-metrizable compact Abelian group H of weight $\geq 2^{\omega}$ contains a dense subgroup that does not determine H. Hence, under the assumption of the continuum hypothesis, every determined compact Abelian group H is metrizable. Subsequently, it was shown in [16] that the result also holds without assuming the continuum hypothesis (see also [8]).

Our goal is to extend the above-noted results to compact groups that are not necessarily Abelian. We now formulate our main results.

Theorem 4.1. *If G is a precompact metrizable group, then $\widehat{G}$ is discrete. Equivalently, every metrizable compact group is determined.*

This extends the aforementioned results from [2,4] to non-Abelian compact metrizable groups.

A certain extension of the Aussenhofer–Chasco theorem to non-Abelian groups is due to Lukács in [21], where he considers the compact-open topology on certain groups of continuous mappings. Theorem 4.1 does not follow from Lukács's results, because in general the Fell topology on the dual object is not reduced to the compact-open topology. (This question is also considered in [13] in the context of uniform spaces.) Let 1_G be the class of the trivial representation.

Theorem 5.1. *If G is a countable infinite precompact non-metrizable group, then 1_G is not an isolated point in $\widehat{G}$.*

Theorem 5.1 also shows that *if G is a countable dense subgroup of a non-metrizable compact group H, then G does not determine H.*

Theorem 5.2. *If H is a non-metrizable compact group, then H has a dense subgroup G such that $\widehat{G}$ is not discrete.*

Together with Theorem 4.1, this shows that *a compact group is determined if and only if it is metrizable.* This extends to non-Abelian compact groups the results given in [6,16,8] for Abelian compact groups.

A group G has property (T) if 1_G is isolated in $\mathcal{R} \cup \{1_G\}$ for every set $\mathcal{R}$ of equivalence classes of unitary representations of G without non-zero invariant vectors. This definition is equivalent to the definition of property (T) in terms of Kazhdan pairs, which we recall below in Section 2. Compact groups have property (T), and we are interested in property (T) for precompact groups.

It follows from Theorem 5.1 that every countable precompact group with property (T) is metrizable. Furthermore, our proof also yields the following result of Wang [26]: if a discrete group G has property (T), then its Bohr compactification bG is metrizable (see Corollary 5.7).

Theorem 6.1. *If G is a countable infinite precompact Abelian group, then G does not have property (T).*

Theorem 6.1 remains true (with the same proof) for every precompact Abelian group whose compact subsets are countable. We do not know whether there exists a non-compact precompact Abelian group with property (T).

2. Preliminaries: Fell topologies and property (T)

All topological groups are assumed to be Hausdorff. For a (complex) Hilbert space $\mathcal{H}$, the unitary group $U(\mathcal{H})$ of all linear isometries of $\mathcal{H}$ is equipped with the strong operator topology. With this topology, $U(\mathcal{H})$ is a topological group. If $\mathcal{H} = \mathbb{C}^n$, we identify $U(\mathcal{H})$ with the *unitary group* $\mathbb{U}(n)$ *of order* n, that is, the compact Lie group of all complex $n \times n$ matrices M for which $M^{-1} = M^*$.

A *unitary representation* ρ of the topological group G is a continuous homomorphism $G \to U(\mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space. A closed linear subspace $E \subseteq \mathcal{H}$ is an *invariant* subspace for $\mathcal{S} \subseteq U(\mathcal{H})$ if $ME \subseteq E$ for all $M \in \mathcal{S}$. If there is a closed subspace E with $\{0\} \subsetneq E \subsetneq \mathcal{H}$ which is invariant for $\mathcal{S}$, then $\mathcal{S}$ is called *reducible*; otherwise, $\mathcal{S}$ is *irreducible*. An *irreducible representation* of G is a unitary representation ρ such that $\rho(G)$ is irreducible.

Two unitary representations $\rho : G \to U(\mathcal{H}_1)$ and $\psi : G \to U(\mathcal{H}_2)$ are *equivalent* if there exists a Hilbert space isomorphism $M : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\rho(x) = M^{-1}\psi(x)M$ for all $x \in G$. The *dual object* of a topological group G is the set $\widehat{G}$ of equivalence classes of irreducible unitary representations of G.

If G is a precompact group, the Peter–Weyl theorem (see [18]) implies that all irreducible unitary representations of G are finite dimensional and determine an embedding of G into the product of unitary groups $\mathbb{U}(n)$.