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The sets of divergence points of self-similar measures are residual

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ABSTRACT

Let μ be a self-similar measure supported on a self-similar set K with the open set condition. For $x \in K$, let $A(D(x))$ be the set of accumulation points of $D_r(x) = \frac{\log \mu(B(x,r))}{\log r}$ as $r \searrow 0$. In this paper, we show that for any closed non-singleton subinterval $I \subset \mathbb{R}$, the set of points x for which the set $A(D(x))$ equals I is either empty or residual.

1. Introduction and statement of results

Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$ $(i = 1, 2, \ldots, N)$ be the contracting similarities with contraction ratios $r_i \in (0, 1)$ and let $(p_1, \ldots, p_N)$ be a probability vector (i.e. $0 < p_i < 1$ for all i and $\sum_{i=1}^N p_i = 1$). Using the framework of [11] we say that K is a self-similar set and μ is a self-similar measure if K is the unique non-empty compact subset of $\mathbb{R}^d$ such that

$$K = \bigcup_i S_i(K),$$

and μ is the unique Borel probability measure on $\mathbb{R}^d$ such that

$$\mu = \sum_i p_i \mu \circ S_i^{-1}.$$

It is well known that the support of μ equals K. We say that the list $(S_1, \ldots, S_N)$ satisfies the open set condition (OSC) (sometimes we also say that the self-similar measure μ satisfies the OSC) if there exists a non-empty, bounded and open set U such that $S_i(U) \subset U$ for all i and $S_i(U) \cap S_j(U) = \varnothing$ for all $i \neq j$.

Multifractal analysis of the self-similar measure μ refers to the study of the fractal geometry of the sets of the points x for which the measure $\mu(B(x, r))$ behaves like r^α for small r. Here $B(x, r)$ is the closed ball of radius r centered at x. That is, we study the sets

$$K_\alpha = \left\{ x \in \mathbb{R}^d : \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right\}, \quad \alpha \geq 0.$$

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The support of the self-similar measure μ has the following natural decomposition:

$$K = \bigcup_{\alpha} K_\alpha \cup \widehat{K},$$

where

$$\widehat{K} = \left\{ x \in K : \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} < \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \right\}.$$

The set $\widehat{K}$ is called the set of divergence points (or irregular set) of the self-similar measure and the elements in the set $\widehat{K}$ are called divergence points. It is well known that the set $\widehat{K}$ has μ-measure zero, see [21]. In other words, the set of divergence points of the self-similar measure is negligible from the measure-theoretical point of view. However, there is an extensive literature showing that the set of divergence points of the self-similar measure and other type irregular sets can be large from the point of view of dimension theory, see [4,5,7,9,14,13,16,18,20,22,23] and references therein. In particular, Barreira and Schmeling [4] and Chen and Xiong [5] showed that the set $\widehat{K}$ has full Hausdorff dimension for $(p_1, \ldots, p_N) \neq (r_1^s, \ldots, r_N^s)$, where s denotes the Hausdorff dimension of K (that is, s is the unique solution of the equation $\sum_i r_i^s = 1$). We remark that Chen and Xiong obtained the above result under the assumption that $(S_1, \ldots, S_N)$ satisfies the strong separation condition (SSC), that is, $S_i(K) \cap S_j(K) = \varnothing$ for all i, j with $i \neq j$. Very recently, using the technique in [8], Xiao, Wu and Gao [23] proved that Chen and Xiong's result remained valid under the OSC. Li and Wu [14] studied the Hausdorff dimensions of some refined subsets of the set $\widehat{K}$ under the OSC, and their results unified the above mentioned results as well as some classical results on the multifractal analysis of the self-similar measure.

The notion of residual set is usually used to describe a set being "large" in a topological sense. Recall that in a metric space X, a set R is called residual if its complement is of the first category. Moreover, in a complete metric space a set is residual if it contains a dense G_δ set, see [19]. We say that a set is large from the topological point of view if it is residual. Recently, some results show that the sets of some kinds of divergence points (or irregular sets) can also be large from the topological point of view. For example, Albeverio, Pratsiovytyi and Torbin [1], Hyde et al. [12] and Olsen [17] proved that the sets of some kinds of divergence points associated with integer expansion are residual. Baek and Olsen [2] discussed the set of extremely non-normal points of self-similar set from the topological point of view. Barreira, Li and Valls [3] proved that the set of divergence points of the Birkhoff averages of a continuous function is residual. Motivated by these results, we show in this paper that the set $\widehat{K}$ is large from the topological point of view. Namely, we prove that $\widehat{K}$ is either empty or residual. In fact, we show that the set of points for which the function $D_r(x)$ has a prescribed set of accumulation points is also residual.

To state our result, we first introduce some notations. For $x \in K$, let $A(D(x))$ denote the set of accumulation points of $D_r(x) := \frac{\log \mu(B(x,r))}{\log r}$ as $r \searrow 0$, that is

$$A(D(x)) = \{y \in (0, +\infty) : \lim_{k\to\infty} D_{r_k}(x) = y \text{ for some } \{r_k\}_k \searrow 0\}.$$

Write $\alpha_{\min} = \min_i \frac{\log p_i}{\log r_i}$ and $\alpha_{\max} = \max_i \frac{\log p_i}{\log r_i}$. It was shown in [14] that the set $A(D(x))$ is either a singleton or a closed subinterval for any $x \in \operatorname{supp} \mu$, and $K_I = \varnothing$ if I is not a closed subinterval of $[\alpha_{\min}, \alpha_{\max}]$.

The following are our main results.

Theorem 1.1. *Assume that $\{S_i\}_{i=1}^N$ satisfies the OSC and $I \subset [\alpha_{\min}, \alpha_{\max}]$ is a closed non-singleton subinterval. Then the set*

$$K_I = \{x \in K : A(D(x)) = I\}$$

is residual if it is not empty.

The following result follows immediately from the above theorem.

Corollary 1.2. *Assume that $\{S_i\}_{i=1}^N$ satisfies the OSC. Then the set $\widehat{K}$ is residual if it is not empty.*

2. Preliminaries

For $n \in \mathbb{N}$, let

$$\Sigma^n = \{1, \ldots, N\}^n.$$

Let $\Sigma^* = \bigcup_n \Sigma^n$ and $\Sigma = \{1, \ldots, N\}^{\mathbb{N}}$. We equip Σ with the distance defined by

$$d(\omega, \omega') = 2^{-n}, \quad \omega = (\omega_i)_{i\in\mathbb{N}},\ \omega' = (\omega'_i)_{i\in\mathbb{N}},$$

where n is the smallest integer such that $\omega_n \neq \omega'_n$. It is well known that (Σ, d) is a compact metric space. For $\omega = (\omega_1 \ldots \omega_n) \in \Sigma^n$, we denote by $|\omega| = n$ the length of ω. For $\omega = (\omega_1 \ldots \omega_n) \in \Sigma^n$ and a positive integer m with $m \le n$, or for $\omega = (\omega_1, \omega_2, \ldots) \in \Sigma$ and a positive integer m, let $\omega|m = (\omega_1 \ldots \omega_m)$. For $\omega = (\omega_1 \ldots \omega_n) \in \Sigma^n$ and $\omega' = (\omega'_1 \ldots \omega'_m) \in \Sigma^m$, we let $\omega\omega' = (\omega_1 \ldots \omega_n \omega'_1 \ldots \omega'_m) \in \Sigma^{n+m}$. Analogously, for $\omega = (\omega_1 \ldots \omega_n) \in \Sigma^n$ and $\omega' = (\omega'_1, \omega'_2, \ldots) \in \Sigma$, we