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Principal Lyapunov exponents and principal Floquet spaces of positive random dynamical systems. II. Finite-dimensional systems

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ABSTRACT

This is the second part in a series of papers concerned with principal Lyapunov exponents and principal Floquet subspaces of positive random dynamical systems in ordered Banach spaces. The current part focuses on applications of general theory, developed in the authors' paper [J. Mierczyński, W. Shen, Principal Lyapunov exponents and principal Floquet spaces of positive random dynamical systems, I, general theory, Trans. Amer. Math. Soc., in press (http://dx.doi.org/10.1090/S0002-9947-2013-05814-X). Preprint available at http://arxiv.org/abs/1209.3475], to positive random dynamical systems on finite-dimensional ordered Banach spaces. It is shown under some quite general assumptions that measurable linear skew-product semidynamical systems generated by measurable families of positive matrices and by strongly cooperative or type-K strongly monotone systems of linear ordinary differential equations admit measurable families of generalized principal Floquet subspaces, generalized principal Lyapunov exponents, and generalized exponential separations.

1. Introduction

This is the second part of a series of several papers. The series is devoted to the study of principal Lyapunov exponents and principal Floquet subspaces of positive random dynamical systems in ordered Banach spaces.

Lyapunov exponents play an important role in the study of asymptotic dynamics of linear and nonlinear random evolution systems. Oseledets obtained in [19] important results on Lyapunov exponents and measurable invariant families of subspaces for finite-dimensional dynamical systems, which are called now the Oseledets multiplicative ergodic theorem. Since then a huge amount of research has been carried out toward alternative proofs of the Oseledets multiplicative ergodic theorem (see [2,9,11,15,18,20,21] and the references contained therein) and extensions of the Oseledets multiplicative theorem for finite dimensional systems to certain infinite dimensional ones (see [2,9,11,13,15,18,20–22,24], and references therein).

The largest finite Lyapunov exponents (or top Lyapunov exponents) and the associated invariant subspaces of both deterministic and random dynamical systems play special roles in the applications to nonlinear systems. Classically, the top finite Lyapunov exponent of a positive deterministic or random dynamical system in an ordered Banach space is called the *principal Lyapunov exponent* if the associated invariant family of subspaces corresponding to it consists of one-dimensional

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subspaces spanned by a positive vector (in such case, invariant subspaces are called the *principal Floquet subspaces*). For more on those subjects see [17].

In the first part of the series, [17], we introduced the notions of generalized principal Floquet subspaces, generalized principal Lyapunov exponents, and generalized exponential separations, which extend the corresponding classical notions. The classical theory of principal Lyapunov exponents, principal Floquet subspaces, and exponential separations for strongly positive and compact deterministic systems is extended to quite general positive random dynamical systems in ordered Banach spaces.

In the present, second part of the series, we consider applications of the general theory developed in [17] to positive random dynamical systems arising from a variety of random mappings and ordinary differential equations. To be more specific, let $((\Omega, \mathfrak{F}, \mathbb{P}), \theta_t)$ be an ergodic metric dynamical system. We investigate positive random matrix models of the form $((U_\omega(n))_{\omega\in\Omega, n\in\mathbb{Z}^+}, (\theta_n)_{n\in\mathbb{Z}})$ (including random Leslie matrix models) (see Section 3), where

$$U_\omega(1)u = \begin{pmatrix} s_{11}(\omega) & s_{12}(\omega) & \cdots & s_{1N}(\omega) \\ s_{21}(\omega) & s_{22}(\omega) & \cdots & s_{2N}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ s_{N1}(\omega) & s_{N2}(\omega) & \cdots & s_{NN}(\omega) \end{pmatrix} u, \quad u \in \mathbb{R}^N, \tag{1.1}$$

$s_{ij}(\omega) \geq 0$ for $i, j = 1, 2, \ldots, N$ and $\omega \in \Omega$; random cooperative systems of ordinary differential equations of the form (see Section 4.1)

$$\dot{u}(t) = A(\theta_t\omega)u(t), \quad \omega \in \Omega,\ t \in \mathbb{R},\ u \in \mathbb{R}^N, \tag{1.2}$$

where

$$A(\omega) = \begin{pmatrix} a_{11}(\omega) & a_{12}(\omega) & \cdots & a_{1N}(\omega) \\ a_{21}(\omega) & a_{22}(\omega) & \cdots & a_{2N}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1}(\omega) & a_{N2}(\omega) & \cdots & a_{NN}(\omega) \end{pmatrix},$$

and $a_{ij}(\omega) \geq 0$ for $i \neq j, i, j = 1, 2, \ldots, N$ and $\omega \in \Omega$; and random type-K monotone systems of ordinary differential equations (see Section 4.2)

$$\dot{u}(t) = B(\theta_t\omega)u(t), \quad \omega \in \Omega,\ t \in \mathbb{R},\ u \in \mathbb{R}^N, \tag{1.3}$$

where for each $\omega \in \Omega$,

$$B(\omega) = \begin{pmatrix} b_{11}(\omega) & b_{12}(\omega) & \cdots & b_{1N}(\omega) \\ b_{21}(\omega) & b_{22}(\omega) & \cdots & b_{2N}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ b_{N1}(\omega) & b_{N2}(\omega) & \cdots & b_{NN}(\omega) \end{pmatrix},$$

and there are $1 \leq k, l \leq N$ such that $k + l = N$, $b_{ij}(\omega) \geq 0$ for $i \neq j$ and $i, j \in \{1, 2, \ldots, k\}$ or $i, j \in \{k+1, k+2, \ldots, k+l\}$, and $b_{ij}(\omega) \leq 0$ for $i \in \{1, \ldots, k\}$ and $j \in \{k+1, \ldots, k+l\}$ or $i \in \{k+1, \ldots, k+l\}$ and $j \in \{1, \ldots, k\}$.

We remark that, biologically, (1.1) describes discrete-time age-structured population models, (1.2) models a system of N species which is cooperative, and (1.3) models a community of N species which can be divided into two subcommunities, one subcommunity consisting of k species and the other consisting of l species, such that the interactions between every pair of species in either subcommunity are cooperative, whereas the interactions between the species belonging to different subcommunities are competitive. The study of (1.1)–(1.3) will provide some basic tool for the study of random discrete-time age-structured nonlinear population models and random cooperative or type-K monotone systems of nonlinear ordinary differential equations. The reader is referred to [4–8,10,14,25,26,28,29], and references therein for the study of discrete-time age-structured population models and time periodic and random cooperative and type-K monotone systems of nonlinear ordinary differential equations.

Under quite general conditions, (1.1), (1.2), generate measurable linear skew-product semidynamical systems on $\Omega \times \mathbb{R}^N$, preserving the natural ordering on $\mathbb{R}^N$ (i.e., the order generated by the cone $(\mathbb{R}^N)^+ = \{\, u = (u_1, \ldots, u_N)^\top : u_i \geq 0, i = 1, \ldots, N \,\}$), and (1.3) generates a measurable linear skew-product semidynamical system on $\Omega \times \mathbb{R}^N$, preserving the type-$K$ ordering on $\mathbb{R}^N$ generated by $(\mathbb{R}^k)^+ \times (\mathbb{R}^l)^- = \{\, u = (u_1, \ldots, u_N)^\top : u_i \geq 0$ for $i = 1, \ldots, k$ and $u_i \leq 0$ for $i = k+1, \ldots, k+l(= N) \,\}$.

Observe that by the following variable change:

$$u_i \mapsto u_i \quad \text{for } i = 1, \ldots, k \quad \text{and} \quad u_i \mapsto -u_i \quad \text{for } i = k+1, \ldots, k+l(= N),$$

the random type-K monotone system (1.3) becomes a random cooperative system of form (1.2) (see Section 4.2 for details). We will therefore focus on the study of (1.1) and (1.2). Applying the general theory developed in Part I [17], we obtain the following results.