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Existence of bounded uniformly continuous mild solutions on $\mathbb{R}$ of evolution equations and their asymptotic behaviour

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ABSTRACT

We prove that $u' = Au + \phi$ has on $\mathbb{R}$ a mild solution $u_\phi \in BUC(\mathbb{R}, X)$ (that is, bounded and uniformly continuous), where A is the generator of a C_0-semigroup on the Banach space X with resolvent satisfying $\|R(it, A)\| = O(|t|^{-\theta})$, $|t| \to \infty$, for some $\theta > \frac{1}{2}$, $\phi \in L^\infty(\mathbb{R}, X)$ and $i\,sp(\phi) \cap \sigma(A) = \emptyset$ (sp = Beurlingspectrum). As a consequence it is shown that if $\mathcal{F}$ is the space of almost periodic, almost automorphic, bounded Levitan almost periodic or one of certain classes of recurrent functions and ϕ as above is such that only $M_h\phi := (1/h)\int_0^h \phi(\cdot + s)\,ds \in \mathcal{F}$ for each $h > 0$, then $u_\phi \in \mathcal{F} \cap BUC(\mathbb{R}, X)$. These results seem new and strengthen several recent theorems.

1. Introduction

In this paper we show the existence of a bounded uniformly continuous mild solution (see (7)) of the evolution equation

$$u' = Au + \phi \quad \text{on } \mathbb{R}, \tag{1}$$

where X is a complex Banach space, $\phi \in L^\infty(\mathbb{R}, X)$ and $A : D(A) \subset X \to X$ is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$, in the non-resonant case, that is if one has

$$i\,sp(\phi) \cap \sigma(A) = \emptyset, \quad \text{where } sp \text{ denotes Beurling spectrum.} \tag{2}$$

As a by-product of our new approach of constructing mild solutions of (1), we are able to extend admissibility with respect to (1) and (2) from subspaces $\mathcal{F}$ of $BUC(\mathbb{R}, X)$ to subspaces $\mathcal{F}$ of $L^\infty(\mathbb{R}, X)$.

Here a linear translation invariant subspace $\mathcal{F} \subset L^\infty(\mathbb{R}, X)$ is called *admissible* with respect to (1) and (2) if for every $\phi \in \mathcal{F}$ satisfying (2), (1) has a mild solution $u_\phi \in \mathcal{F}$.

The definitions of admissibility vary, see [16, p. 126], [15, p. 167], [21, Definition 11.3, p. 287, 306], [20, p. 401], [17, p. 248].

We note that without (2) a bounded mild solution of (1) may not exist even in the simplest case when $A = 0$.

In [4, Theorem 6.5(ii)] using results on the operator equation $AX - XB = C$ of [19] it has been shown that $BUC(\mathbb{R}, X)$ is admissible if A is the generator of a holomorphic C_0-semigroup T with $\sup_{t>0} \|T(t)\| < \infty$. Using the spectral properties of the sum of commuting operators from [1, Theorem 7.3] a new approach to admissibility has been given in [19,20] and [17, results in Section 3] for $\mathcal{F} \subset BUC(\mathbb{R}, X)$ if either $sp(\phi)$ is compact or $(T(t))_{t\geq 0}$ is holomorphic or $(T(t))_{t\geq 0}$ admits exponential dichotomy. In [11] this method is extended to the case ϕ is Bochner almost automorphic.

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This paper contains 6 sections. In Section 2, we give notation and definitions. In Section 3, we collect and extend some preliminary results needed subsequently. In Section 4, we construct a Green function $G \in L^1(\mathbb{R}, L(X))$ (Lemma 4.4, Proposition 4.5). In our main result (Theorem 5.2) below we prove the existence of a bounded uniformly continuous mild solution u_ϕ on $\mathbb{R}$ of the form $u_\phi = G * \phi$ for any $\phi \in L^\infty(\mathbb{R}, X)$ satisfying (2), when A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ with resolvent satisfying $\|R(it, A)\| = O(|t|^{-\theta})$, $|t| \to \infty$, for some $\theta > \frac{1}{2}$. Note that $(T(t))_{t\geq 0}$ holomorphic is the case $\theta = 1$. This gives new classes admissible with respect to (1) and (2).

If in Theorem 5.2 additionally $\sup_{t>0} \|T(t)\| < \infty$, then for each $x \in X$ the unique mild solution of the Cauchy problem $u(0) = x$ on $[0, \infty)$ is in $BUC(\mathbb{R}_+, X)$. Comparing this with the Non-resonance Theorem 5.6.5 of [2], our result is a 3-fold extension. It gives solutions on $\mathbb{R}$ instead of $[0, \infty)$ and $(T(t))_{t\geq 0}$ need not be bounded or holomorphic. Theorem 5.6.5 of [2] is in one sense more general since it uses the (smaller) half-line spectrum instead of the Beurling spectrum used in (2); however in the important cases of almost periodic, almost automorphic, Levitan almost periodic or recurrent functions these two spectra coincide by [8, Example 3.8], [9, Corollary 5.2].

In Section 6 (Lemma 6.1), we prove admissibility of classes satisfying (30)–(33). In particular, this generalizes a result of [11, Theorem 4.5] in several directions: The semigroup $(T(t))_{t\geq 0}$ need not be holomorphic, ϕ must only be a "mean" ((29)) Veech almost automorphic function [6, (3.3)], the solution u_ϕ is in addition bounded and uniformly continuous. Examples 6.2 show that $\mathcal{F}$ may contain not necessarily compact or uniformly continuous elements; these seem not to be treatable by the methods used in [19,4,20,17,11]. In Example 5.4 and Remarks 6.4 we discuss the sharpness of our results.

2. Notation and definitions

In the following $\mathbb{R}_+ = [0, \infty)$, $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $(X, \|\cdot\|)$ is a complex Banach space, and $L(X)$ is the Banach algebra of bounded linear maps $B : X \to X$ with operator norm $\|B\|$.

$\mathcal{D}(\mathbb{R})$ denotes the set of Schwartz's complex valued infinitely differentiable functions on $\mathbb{R}$ with compact support and $\mathcal{S}(\mathbb{R})$ stands for Schwartz's complex valued infinitely differentiable functions on $\mathbb{R}$ with rapidly decreasing derivatives.

$$C(\mathbb{J}, X),\ BC(\mathbb{J}, X),\ BUC(\mathbb{J}, X),\ C_0(\mathbb{J}, X) \text{ and } C^\infty(\mathbb{R}, X) \text{ with } \mathbb{J} \in \{\mathbb{R}_+, \mathbb{R}\}$$

are the spaces of continuous, bounded continuous, bounded uniformly continuous, continuous vanishing at infinity and infinitely differentiable functions.

We recall that $f \in C(\mathbb{R}, X)$ is called Bochner almost automorphic [26, p. 66] if for any sequence $(t_n') \subset \mathbb{R}$ there exists a subsequence (t_n) such that

$$\lim_{n\to\infty} f(t + t_n) =: g(t), \quad t \in \mathbb{R} \quad \text{and} \quad \lim_{n\to\infty} g(t - t_n) = f(t), \quad t \in \mathbb{R}. \tag{3}$$

Similarly, $f \in C(\mathbb{R}, X)$ is called Veech almost automorphic if for any net $(t_i')_{i\in I'} \subset \mathbb{R}$ there exists a subnet $(t_i)_{i\in I}$ such that (3) is satisfied. If in (3) the first limit exist uniformly on $\mathbb{R}$, then f is called almost periodic. $f \in BC(\mathbb{R}, X)$ belongs to $PAP_0(\mathbb{R}, X)$ if it satisfies $\lim_{t\to\infty} \frac{1}{t} \int_{-t}^{t} \|f(s)\|\, ds = 0$ [27, p. 56].

The Banach spaces of almost periodic [2, p. 289], [26, p. 20], [27, p. 2, 9], Bochner almost automorphic [26, p. 66], [11, p. 3292], Veech almost automorphic [6, p.430], pseudo almost periodic and pseudo almost automorphic [27, p. 57], [6, p. 424] functions are denoted by

$$AP = AP(\mathbb{R}, X), \qquad AA = AA(\mathbb{R}, X), \qquad VAA = VAA(\mathbb{R}, X),$$
$$PAP = AP(\mathbb{R}, X) \oplus PAP_0(\mathbb{R}, X) \quad \text{and} \quad PAA = AA(\mathbb{R}, X) \oplus PAP_0(\mathbb{R}, X).$$

For $f \in L^1_{loc}(\mathbb{J}, X)$, $(Pf)(t) :=$ Bochner integral $\int_0^t f(s)\, ds$, for $h > 0$ the mollifier $M_h f(t) := \frac{1}{h} \int_0^h f(t + s)\, ds$ and $f_a(t) =$ translate $f(a + t)$ where defined, a real. For $f \in L^1(\mathbb{R}, X)$, the Fourier transform is defined by $\widehat{f}(\lambda) = \int_{\mathbb{R}} f(t) e^{-i\lambda t}\, dt$, $\lambda \in \mathbb{R}$, and $\widehat{\mathcal{S}}(\mathbb{R}) = \{\widehat{\varphi} : \varphi \in \mathcal{S}(\mathbb{R})\}$.

sp denotes the *Beurling spectrum*, $sp(\phi) = sp_{\{0|R\}}(\phi)$ as defined for example in [8, (3.2), (3.3)] with $S = \phi \in L^\infty(\mathbb{R}, X)$, $V = L^1(\mathbb{R}, \mathbb{C})$, $\mathcal{A} = \{0|\mathbb{R}\}$:

$$sp_{\{0|\mathbb{R}\}}(\phi) := \{\omega \in \mathbb{R} : f \in L^1(\mathbb{R}, \mathbb{C}), \phi * f = 0 \text{ imply } \widehat{f}(\omega) = 0\}. \tag{4}$$

sp_B is defined in [2, p. 321], and sp_C is the Carleman spectrum [2, p. 293/317].

For a linear operator $A : D(A) \subset X \to X$ with domain $D(A)$, $\sigma(A)$ denotes the spectrum of A and $R(\cdot, A) : \mathbb{C} \setminus \sigma(A) \to L(X)$ denotes the resolvent of A [2, Appendix B, p. 462]. The k^{th} derivative of R, $R(t) := R(it, A)$, is denoted by $R^{(k)}$, where $k \in \mathbb{N}$ and $R^{(0)} = R$ (see (15)).

3. Preliminaries

We study solutions of the inhomogeneous abstract evolution equation

$$\frac{du(t)}{dt} = Au(t) + \phi(t), \quad t \in \mathbb{J}, \tag{5}$$

where the linear operator $A : D(A) \subset X \to X$ is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on the complex Banach space X, $\mathbb{J} \in \{\mathbb{R}_+, \mathbb{R}\}$ and $\phi \in L^1_{loc}(\mathbb{J}, X)$.