J. Math. Anal. Appl. 404 (2013) 570–578

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

On spherical expansions of smooth $\mathbf{SU}(n)$-zonal functions on the unit sphere in $\mathbb{C}^n$

Agata Bezubik[a], Aleksander Strasburger[b,*]

[a] Institute of Mathematics, University of Białystok, Akademicka 2, 15-267 Białystok, Poland

[b] Department of Applied Mathematics, Warsaw University of Life Sciences — SGGW, Nowoursynowska 166, 02-787 Warszawa, Poland

ARTICLE INFO

Article history:
Received 14 August 2012
Available online 22 March 2013
Submitted by D. Khavinson

Keywords:
Laplace operator
Bihomogeneous spherical harmonics
Zonal harmonic polynomials
Jacobi polynomials
Funk–Hecke formula
Poisson–Szegö kernel

ABSTRACT

We give a self-contained presentation of a novel approach to the spherical harmonic expansions of smooth zonal functions defined on the unit sphere in $\mathbb{C}^n$. The main new result is a formula expressing the coefficients of the expansion in terms of the Taylor coefficients of the profile function. This enables us to give a new form of the classical Funk–Hecke formula for the case of complex spheres. As another application we give a new derivation the spherical harmonic expansion for the Poisson–Szegö kernel for the unit ball in $\mathbb{C}^n$ obtained originally by Folland.

1. Introduction

Given a function f on the unit sphere S in the Euclidean space $\mathbb{R}^d$, $d > 1$, the question of determining its spherical harmonic expansion is naturally solved by resorting to integration — i.e. using an integral formula for the harmonic projection. The aim of this paper, and its preceding companion [4], is to show that at least for suitably regular functions, an answer in a form of a closed expression can be obtained also by means of differential calculus.

The case treated here concerns odd dimensional spheres, regarded as unit spheres in the complex space $\mathbb{C}^n$. It goes beyond the classical theory of spherical harmonic expansions in the sense that it requires the use of an orthogonal system in two variables — more specifically, of an orthogonal system on the unit disc constructed in terms of the so called disc polynomials related to the classical Jacobi polynomials, cf. e.g. [12,17]. However the theory of expansions in terms of orthogonal systems in several variables is relevant for us only so far as being the tool for parametrizing the zonal functions on the unit sphere in $\mathbb{C}^n$.

The majority of this paper is based on results from the doctoral dissertation of one of the present authors (Agata Bezubik, cf. [2]) concerned with spherical harmonic expansions of zonal functions on the unit spheres. The thesis was submitted in June 2010, almost at the same time as an article [13] of V.A. Menegatto, A.P. Peron, and C.P. Oliveira was published. Acknowledging an inspiration of our earlier paper [4], the authors of that paper formulate an expansion theorem (Theorem 2.3 of their paper) with an almost identical content as our Theorem 2, however their approach is substantially different from ours. Moreover, the original version of their paper contains a number of inaccuracies regarding various coefficients appearing in the formulae stated therein, subsequently corrected in an Erratum [14] (to appear).

In order to keep the paper reasonably self-contained and to introduce the necessary notation we devote a few pages at the beginning to present essential facts concerning harmonic analysis on unit spheres, particularly in the complex case.

* Corresponding author.
E-mail addresses: agatab@math.uwb.edu.pl (A. Bezubik), aleksander_strasburger@sggw.pl (A. Strasburger).

0022-247X/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jmaa.2013.03.035

The main result concerning expansions of zonal functions is stated as the Theorem 2 and applied to obtain a new form of the Funk–Hecke formula in the Section 3. As its main application, Folland's formula for the expansion of the Poisson–Szegö kernel for the unit ball (Theorem 3) is rederived from the spherical harmonic expansions formulae (24)–(25) at the end of the paper.

2. Preliminaries

We shall be concerned with problems in analysis on the unit sphere $S = \{z \in \mathbb{C}^n \mid |z|^2 = 1\}$ in the complex n-space $\mathbb{C}^n$. The latter is equipped with the usual norm $|z| = \sqrt{(z|z)}$ derived from the Hermitian inner product $(z|w) = \sum_{j=1}^n z_j \bar{w}_j$. We shall write $r^2 = (z|z)$ for the square of the norm. Given $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we set $z_j = x_j + iy_j$, with real x_j, y_j and $i = \sqrt{-1}$, for $j = 1, \dots, n$. This gives the standard identification of $\mathbb{C}^n$ with the real Cartesian space $\mathbb{R}^{2n}$ by $(z_1, \dots, z_n) \leftrightarrow (x_1, \dots, x_n, y_1, \dots, y_n)$, which in turn enables us to express objects attached to the Euclidean space $\mathbb{R}^{2n}$ in terms of complex coordinates.

In particular, polynomials on $\mathbb{R}^{2n}$ can be written as polynomials of the complex coordinates z_j and their conjugates $\bar{z}_j$, which will be indicated by writing $P(x, y) = P(z, \bar{z})$ for an element of the polynomial algebra $\mathcal{P}(\mathbb{R}^{2n}) \simeq \mathcal{P}(\mathbb{C}^n)$. Further, using the standard notation for complex partials $\partial_j = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$, $\bar{\partial}_j = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$ with $j = 1, \dots, n$, the Euclidean Laplacian Δ on $\mathbb{R}^{2n}$ can be expressed in the complex form by

$$\Delta = 4\sum_{j=1}^n \partial_j \bar{\partial}_j. \tag{1}$$

The special unitary group $\mathbf{SU}(n)$ acts transitively on S with the isotropy group of a point $\eta \in S$ isomorphic to the group $\mathbf{SU}(n-1)$. Thus, along with the usual representation $S = S^{2n-1} = \mathbf{SO}(2n)/\mathbf{SO}(2n-1)$ of S as a homogeneous space of the orthogonal group $\mathbf{SO}(2n)$, we can identify S with the homogeneous space $\mathbf{SU}(n)/\mathbf{SU}(n-1)$. The choice of reference point $\eta \in S$ is not relevant, since isotropy groups are conjugate and it will be of some advantage not to fix this point by any specific choice. The sphere S is equipped with the normalized Euclidean surface measure $d\sigma_{2n-1} = d\sigma$ $\left(\text{i.e. } \int_S d\sigma = 1\right)$, which is $\mathbf{O}(2n)$-invariant, as well as $\mathbf{SU}(n)$-invariant.

A departure point for analysis on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ is the orthogonal decomposition

$$L^2(S^{d-1}, d\sigma) = \bigoplus_{l=0}^{\infty} H^l, \tag{2}$$

where H^l denotes the space of restrictions to the sphere of harmonic (i.e. annihilated by the Laplacian) and homogeneous of degree l polynomials on $\mathbb{R}^d$. A key property of those spaces is the irreducibility under the action of the orthogonal group $\mathbf{SO}(d)$.

For the case of the unit sphere in $\mathbb{C}^n$, the main role is assigned to the group $\mathbf{SU}(n)$, which requires a refinement of the decomposition (2) obtained by employing bihomogeneous polynomials. Recall that given a pair of non-negative integers p, q, a polynomial $P(z, \bar{z})$ on $\mathbb{C}^n$ is said to be bihomogeneous of degree (p, q) if $P(\lambda z, \bar{\lambda}\bar{z}) = \lambda^p \bar{\lambda}^q P(z, \bar{z})$ for all $z \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. The space of all such polynomials is denoted by $\mathcal{P}^{(p,q)} = \mathcal{P}^{(p,q)}(\mathbb{C}^n)$. We notice that $\mathcal{P}^{(p,0)}$ contains only holomorphic polynomials, while $\mathcal{P}^{(0,q)}$ — antiholomorphic ones. We shall write $\mathcal{H}^{(p,q)} = \mathcal{H}^{(p,q)}(\mathbb{C}^n) = \ker \Delta \cap \mathcal{P}^{(p,q)}(\mathbb{C}^n)$ for the space of *(solid) harmonics of bidegree* (p, q) on $\mathbb{C}^n$. We set $H^{(p,q)} = \mathcal{H}^{(p,q)}|_S$ — the space of restrictions to S of solid harmonics from $\mathcal{H}^{(p,q)}$, and call *surface harmonics* its elements. All these spaces are invariant under the natural action of $\mathbf{SU}(n)$, while $\mathcal{H}^{(p,q)}$ (and its isomorphic image $H^{(p,q)}$) are moreover irreducible. One has

$$\dim \mathcal{H}^{(p,q)} = \frac{(n+p+q-1)(n-2+p)!(n-2+q)!}{p!\,q!\,(n-1)!\,(n-2)!}.$$

Clearly, (p, q) bihomogeneous polynomials are homogeneous of degree $p+q$ in the real sense and there are direct sum decompositions

$$\mathcal{P}^l(\mathbb{C}^n) = \bigoplus_{p+q=l} \mathcal{P}^{(p,q)}(\mathbb{C}^n), \qquad \mathcal{H}^l(\mathbb{C}^n) = \bigoplus_{p+q=l} \mathcal{H}^{(p,q)}(\mathbb{C}^n), \tag{3}$$

of the spaces of homogeneous of degree l, resp. homogeneous and harmonic, polynomials on $\mathbb{C}^n$. Consequently the decomposition (2) can be refined to a $\mathbf{SU}(n)$-irreducible orthogonal decomposition

$$L^2(S, d\sigma) = \bigoplus_{p,q=0}^{\infty} H^{(p,q)}. \tag{4}$$

Analogous to the real case, there is a decomposition of bihomogeneous polynomials into harmonic components. In fact, $\Delta : \mathcal{P}^{(p,q)}(\mathbb{C}^n) \to \mathcal{P}^{(p-1,q-1)}(\mathbb{C}^n)$ is surjective with the kernel $\mathcal{H}^{(p,q)}(\mathbb{C}^n)$ and consequently there is a $\mathbf{SU}(n)$-invariant decomposition $\mathcal{P}^{(p,q)}(\mathbb{C}^n) = \bigoplus_{k=0}^{\min(p,q)} r^{2k}\mathcal{H}^{(p-k,q-k)}(\mathbb{C}^n)$. The explicit formulae for this decomposition can be deduced from