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Ground state solutions for superlinear elliptic systems on $\mathbb{R}^N$ ☆

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ABSTRACT

This paper is concerned with the existence of the ground state solutions for the following superlinear elliptic system of Hamiltonian type,

$$\begin{cases} -\Delta u + V(x)u = g(x, v) & \text{in } \mathbb{R}^N, \\ -\Delta v + V(x)v = f(x, u) & \text{in } \mathbb{R}^N, \\ u(x) \to 0 \quad \text{and} \quad v(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where $V \in C(\mathbb{R}^N, \mathbb{R})$ is periodic in $x_1, x_2, \ldots, x_N$. We assume that 0 lies in a gap of the spectrum $-\Delta + V$, and f and g are both superlinear at 0 and infinity but they have different increasing rates at infinity. By proving all Cerami sequences for the energy functional are bounded, existence of a ground state solution is obtained.

1. Introduction

In this paper, we study the following superlinear elliptic system

$$\begin{cases} -\Delta u + V(x)u = g(x, v) & \text{in } \mathbb{R}^N, \\ -\Delta v + V(x)v = f(x, u) & \text{in } \mathbb{R}^N, \\ u(x) \to 0 \quad \text{and} \quad v(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{P}$$

where $N \geq 3$, $V \in C(\mathbb{R}^N, \mathbb{R})$ is a general periodic function and $f(x, t),\ g(x, t) \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$.

For a similar problem on a bounded domain we refer the reader to [5,8]. The problem (P) in the whole space $\mathbb{R}^N$ was considered recently in some works. Most of them focused on the case $V \equiv 1$, in such case one can work on the radially symmetric function space which possesses compact embedding, see, for example, [3,6,11–13]. The dual variational method is involved in some works, see for instance [1,2,14,17] and the references therein. Recently, (P) with general periodic potential was considered in [19–22].

Motivated by these works, we continue to consider the periodic case, and we make the following assumption on the potential V:

(S_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in each x_i for $i = 1, \ldots, N$, and 0 lies in a gap of $\sigma(S)$, where $S := -\Delta + V$.

There are at least three difficulties in our problem. First, there is a lack of the compactness of the Sobolev embedding since the domain is the whole $\mathbb{R}^N$. Second, the variational setting for our problem is more complex and different with the

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case where $V = 1$ since the potential V is a general periodic function. Third, the energy function is strongly indefinite and it has more complex geometry structure than functions which have mountain pass structure. Therefore, working on the radially symmetric function space or using the dual variational method cannot be applied here.

We suppose that f and g are "superlinear" at zero and infinity and "subcritical" at infinity. Our precise assumptions on f and g are the following.

(S_2) $f, g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ are 1-periodic in each x_i for $i = 1, \ldots, N$, and there is a constant $C > 0$ such that

$$|f(x,t)| \le C(1+|t|^p) \quad \text{and} \quad |g(x,t)| \le C(1+|t|^q)$$

for all (x, t), where $p,\ q > 1$ satisfy

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}. \tag{1.1}$$

(S_3) $f(x,t) = o(|t|)$ and $g(x,t) = o(|t|)$ uniformly in x as $|t| \to 0$.

(S_4) $\frac{F(x,t)}{t^2} \to \infty$ and $\frac{G(x,t)}{t^2} \to \infty$ uniformly in x as $|t| \to \infty$, where $F(x,t) = \int_0^t f(x,s)ds$, $G(x,t) = \int_0^t g(x,s)ds$.

(S_5) $F(x,t) \ge 0$, $G(x,t) \ge 0$, $\mathcal{F}(x,t) = \frac{1}{2}f(x,t)t - F(x,t) \ge 0$ and $\mathcal{G}(x,t) = \frac{1}{2}g(x,t)t - G(x,t) \ge 0$.

(S_6) For $z = (u, v)$, $w = (w^{(1)}, w^{(2)})$, $s \ge -1$ with $sz + w \ne 0$, there exists a function $W(x) \in L^1(\mathbb{R}^N)$, there holds

$$f(x,u)\left[s\left(\frac{s}{2}+1\right)u + (1+s)w^{(1)}\right] + F(x,u) - F(x, u+w^{(1)}) \le W(x) \tag{1.2}$$

and

$$g(x,v)\left[s\left(\frac{s}{2}+1\right)v + (1+s)w^{(2)}\right] + G(x,v) - G(x, v+w^{(2)}) \le W(x). \tag{1.3}$$

Remark 1. (S_6) holds if the following (S_6') holds, see Lemma 3.2 in [9].

Very recently, under the hypothesis (S_1)–(S_4) and

(S_6') $f(x,t)/|t|$ is increasing on $\mathbb{R} \setminus \{0\}$,

the ground state solution for

$$\begin{cases} -\Delta u + V(x)u = f(x,u) & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.4}$$

was obtained in [9], the key point is that the author shows that all Cerami sequences for the energy functional are bounded, the results generalized the earlier works in [10,15,18]. We should point out in the above three works the authors all need stronger assumption

(S_6'') $t \mapsto \frac{f(x,t)}{|t|}$ is strictly increasing on $\mathbb{R} \setminus \{0\}$

since the Nehari manifold method or the monotony trick were involved.

In this paper, we consider the existence of the ground state solution for the system (P). Our main result is the following

Theorem 1.1. *Suppose that (S_1)–(S_6) are satisfied, then (P) has a ground state.*

The paper is organized as following. In Section 2, we set up the framework in which we study the variational problem associated with (P). In Section 3, we show that the energy functional has linking structure. In Section 4, we will prove Theorem 1.1.

2. Variational setting

The natural functional associated with (P) is

$$I(u,v) = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv)dx - \int_{\mathbb{R}^N} (F(x,u) + G(x,v))dx \tag{2.1}$$

on the natural space $E := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. But to ensure that I is a C^1 functional one should assume that $1 < p, q \le \frac{N+2}{N-2}$ which is stronger than (1.1). Following [20,22], we use suitable fractional power of some self-adjoint operator on Sobolev space to define the energy functional, the idea can be tracked to [5,8].

Denote c or c_i stands for different positive constants. By (S_1), we know that

$$\bar{\Lambda} := \sup[\sigma(S) \cap (-\infty, 0)] < 0 < \underline{\Lambda} := \inf[\sigma(S) \cap (0, \infty)].$$