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A Cesàro average of Hardy–Littlewood numbers

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ABSTRACT

Let Λ be the von Mangoldt function and $r_{HL}(n) = \sum_{m_1+m_2^2=n} \Lambda(m_1)$ be the counting function for the Hardy–Littlewood numbers. Let N be a sufficiently large integer. We prove that

$$\begin{aligned}\sum_{n\le N} r_{HL}(n)\frac{(1-n/N)^k}{\Gamma(k+1)} &= \frac{\pi^{1/2}}{2}\frac{N^{3/2}}{\Gamma(k+5/2)} - \frac{1}{2}\frac{N}{\Gamma(k+2)}\\ &\quad - \frac{\pi^{1/2}}{2}\sum_{\rho}\frac{\Gamma(\rho)}{\Gamma(k+3/2+\rho)}N^{1/2+\rho}\\ &\quad + \frac{1}{2}\sum_{\rho}\frac{\Gamma(\rho)}{\Gamma(k+1+\rho)}N^{\rho}\\ &\quad + \frac{N^{3/4-k/2}}{\pi^{k+1}}\sum_{\ell\ge1}\frac{J_{k+3/2}(2\pi\ell N^{1/2})}{\ell^{k+3/2}}\\ &\quad - \frac{N^{1/4-k/2}}{\pi^{k}}\sum_{\rho}\Gamma(\rho)\frac{N^{\rho/2}}{\pi^{\rho}}\sum_{\ell\ge1}\frac{J_{k+1/2+\rho}(2\pi\ell N^{1/2})}{\ell^{k+1/2+\rho}}\\ &\quad + \mathcal{O}_k(1),\end{aligned}$$

for $k > 1$, where ρ runs over the non-trivial zeros of the Riemann zeta-function $\zeta(s)$ and $J_\nu(u)$ denotes the Bessel function of complex order ν and real argument u.

1. Introduction

We continue our recent work on additive problems with prime summands. In [8], we studied the *average* number of representations of an integer as a sum of two primes, whereas in [9] we considered individual integers. In this paper, we study a Cesàro weighted *explicit* formula for Hardy–Littlewood numbers (integers that can be written as a sum of a prime and a square) and the goal is similar to the one in [10], that is, we want to obtain an asymptotic formula with the expected main term and one or more terms that depend explicitly on the zeros of the Riemann zeta-function. Letting

$$r_{HL}(n) = \sum_{m_1+m_2^2=n} \Lambda(m_1), \tag{1}$$

the main result of the paper is the following theorem.

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Theorem 1. *Let N be a sufficiently large integer. We have*

$$\sum_{n\leq N} r_{HL}(n)\frac{(1-n/N)^k}{\Gamma(k+1)} = \frac{\pi^{1/2}}{2}\frac{N^{3/2}}{\Gamma(k+5/2)} - \frac{1}{2}\frac{N}{\Gamma(k+2)} - \frac{\pi^{1/2}}{2}\sum_{\rho}\frac{\Gamma(\rho)}{\Gamma(k+3/2+\rho)}N^{1/2+\rho} + \frac{1}{2}\sum_{\rho}\frac{\Gamma(\rho)}{\Gamma(k+1+\rho)}N^{\rho} + \frac{N^{3/4-k/2}}{\pi^{k+1}}\sum_{\ell\geq 1}\frac{J_{k+3/2}(2\pi\ell N^{1/2})}{\ell^{k+3/2}} - \frac{N^{1/4-k/2}}{\pi^k}\sum_{\rho}\Gamma(\rho)\frac{N^{\rho/2}}{\pi^{\rho}}\sum_{\ell\geq 1}\frac{J_{k+1/2+\rho}(2\pi\ell N^{1/2})}{\ell^{k+1/2+\rho}} + \mathcal{O}_k(1),$$

for $k > 1$, where ρ runs over the non-trivial zeros of the Riemann zeta-function $\zeta(s)$ and $J_\nu(u)$ denotes the Bessel function of complex order ν and real argument u.

Similar averages of arithmetical functions are common in the literature; see Chandrasekharan and Narasimhan [2] and Berndt [1] who built on earlier classical works (Hardy, Landau, Walfisz and others). In their setting, the generalized Dirichlet series associated to the arithmetical function satisfies a suitable functional equation and this leads to an asymptotic formula containing Bessel functions of real order and argument. In our case, we have no functional equation, and, as far as we know, it is the first time that Bessel functions with complex order arise in a similar problem. Moreover, from a technical point of view, the estimates of such Bessel functions are harder to perform than the ones already present in the Number Theory literature since the real argument and the complex order are both unbounded while, in previous papers, either the real order or the argument is bounded.

The method we will use in this additive problem is based on a formula due to Laplace [11], namely

$$\frac{1}{2\pi i}\int_{(a)} v^{-s}e^{v}\,\mathrm{d}v = \frac{1}{\Gamma(s)}, \tag{2}$$

where $\Re(s) > 0$ and $a > 0$; see Formula 5.4(1) on p. 238 of [4]. In the following, we will need the general case of (2) which can be found in de Azevedo Pribitkin [3], formulas (8) and (9):

$$\frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{iDu}}{(a+iu)^s}\,\mathrm{d}u = \begin{cases}\dfrac{D^{s-1}e^{-aD}}{\Gamma(s)} & \text{if } D>0,\\ 0 & \text{if } D<0,\end{cases} \tag{3}$$

which is valid for $\sigma = \Re(s) > 0$ and $a \in \mathbb{C}$ with $\Re(a) > 0$, and

$$\frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{(a+iu)^s}\,\mathrm{d}u = \begin{cases}0 & \text{if } \Re(s)>1,\\ 1/2 & \text{if } s=1,\end{cases} \tag{4}$$

for $a \in \mathbb{C}$ with $\Re(a) > 0$. Formulas (3)–(4) enable us to write averages of arithmetical functions by means of line integrals as we will see in Section 2.

We will also need Bessel functions of complex order ν and real argument u. For their definition and main properties we refer to Watson [14]. In particular, Eq. (8) on p. 177 gives the Sonine representation:

$$J_\nu(u) := \frac{(u/2)^\nu}{2\pi i}\int_{(a)} s^{-\nu-1}e^{s}e^{-u^2/4s}\,\mathrm{d}s, \tag{5}$$

where $a > 0$ and $u, \nu \in \mathbb{C}$ with $\Re(\nu) > -1$. We will also use the Poisson integral formula

$$J_\nu(u) := \frac{2(u/2)^\nu}{\pi^{1/2}\Gamma(\nu+1/2)}\int_0^1 (1-t^2)^{\nu-1/2}\cos(ut)\,\mathrm{d}t, \tag{6}$$

which holds for $\Re(\nu) > -1/2$ and $u \in \mathbb{C}$ (see Eq. (3) on p. 48 of [14]). An asymptotic estimate we will need is

$$J_\nu(u) = \left(\frac{2}{\pi u}\right)^{1/2}\cos\left(u - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + \mathcal{O}_{|\nu|}\left(u^{-5/2}\right), \tag{7}$$

which follows from Eq. (1) on p. 199 of Watson [14].

As in [10], we combine this approach with line integrals with the classical methods dealing with infinite sums over primes, exploited by Hardy and Littlewood (see [6,7]) and by Linnik [12]. The main difference here is that the problem naturally involves the modular relation for the complex theta function (see Eq. (9)). The presence of the Bessel functions in our statement strictly depends on such modularity relation. It is worth mentioning that it is not clear how to get such "modular" terms using the finite sum approach for the Hardy–Littlewood function $r_{HL}(n)$.