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Composition operators acting on weighted Dirichlet spaces

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ABSTRACT

We study composition operators acting on weighted Dirichlet spaces. We obtain estimates for the essential norm, describe the membership in Schatten–Von Neumann ideals and characterize the composition operators with closed range.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane, and let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. The function φ induces a composition operator C_φ acting on $H(\mathbb{D})$, the space of all analytic functions on $\mathbb{D}$, by the formula

$$C_\varphi f(z) = f(\varphi(z)).$$

It is an interesting problem to describe the operator properties of C_φ in terms of the function properties of the symbol φ when the operator C_φ acts on several spaces of analytic functions in $\mathbb{D}$. In this paper, we are going to study the composition operator C_φ acting on weighted Dirichlet spaces $\mathcal{D}_\alpha$, $\alpha > 0$, so let us proceed to introduce these spaces.

For $\alpha \geq 0$, the Dirichlet type space $\mathcal{D}_\alpha$ consists of those analytic functions f on $\mathbb{D}$ with

$$\|f\|_{\mathcal{D}_\alpha} = \left(|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA_\alpha(z) \right)^{1/2} < \infty,$$

where

$$dA_\alpha(z) = (1+\alpha)\,(1-|z|^2)^\alpha \, dA(z),$$

and $dA(z) = \frac{1}{\pi}\, dx\, dy$ is the normalized area measure on $\mathbb{D}$. Observe that $f \in \mathcal{D}_\alpha$ if and only if f' belongs to the Bergman space A^2_α. Let $\alpha > -1$, the weighted Bergman space A^2_α consists of those functions $f \in H(\mathbb{D})$ with

$$\|f\|_{A^2_\alpha} = \left(\int_{\mathbb{D}} |f(z)|^2 \, dA_\alpha(z) \right)^{1/2} < \infty.$$

It is well known that $\mathcal{D}_1 = H^2$, the classical Hardy space, and $\mathcal{D}_\alpha = A^2_{\alpha-2}$ if $\alpha > 1$. The results we are going to obtain about composition operators on the spaces $\mathcal{D}_\alpha$ are well known for the Hardy and Bergman spaces. Therefore we will focus on the case $0 < \alpha < 1$ (the reason why we exclude the case $\alpha = 0$ will be explained later). We are going to estimate the essential norm of the composition operator acting on $\mathcal{D}_\alpha$, obtain a description of the membership in the Schatten–Von Neumann ideal S_p of the composition operator on $\mathcal{D}_\alpha$, and finally, we characterize the composition operators with closed range. All of this will be accomplished with the aid of the generalized Nevanlinna counting function, which we study in the next section.

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2. The generalized Nevanlinna counting function

Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic. The (classical) *Nevanlinna counting function* of φ is

$$N_\varphi(z) = \sum_{w:\varphi(w)=z} \log \frac{1}{|w|}, \quad z \in \mathbb{D} \setminus \{\varphi(0)\},$$

where the sum is interpreted as being zero if $z \notin \varphi(\mathbb{D})$. An important property of the counting function N_φ is that, being not necessarily subharmonic, it satisfies the submean value property (see [22] or [24]).

Submean value property for N_φ. Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic. Then

$$N_\varphi(z) \le \frac{1}{|B|} \int_B N_\varphi(w)\, dA(w),$$

where $z \in \mathbb{D} \setminus \{\varphi(0)\}$ and B is any euclidean disk centered at z contained in $\mathbb{D} \setminus \{\varphi(0)\}$. Here $|B|$ denotes the area of the disk B.

Let $\alpha > 0$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. The *generalized Nevanlinna counting function* of φ is

$$N_{\varphi,\alpha}(z) = \sum_{w:\varphi(w)=z} (1-|w|^2)^\alpha, \quad z \in \mathbb{D} \setminus \{\varphi(0)\},$$

where, as before, the last sum is interpreted as being zero if $z \notin \varphi(\mathbb{D})$.

The following change of variables formula can be found, for example, in [4, Proposition 2.1] or [6, Theorem 2.32].

Change of variables formula. If f is nonnegative on $\mathbb{D}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic, then

$$\int_{\mathbb{D}} f\big(\varphi(z)\big)\, |\varphi'(z)|^2\, (1-|z|^2)^\alpha\, dA(z) = \int_{\mathbb{D}} f(z)\, N_{\varphi,\alpha}(z)\, dA(z).$$

For $a \in \mathbb{D}$, denote by σ_a the disk automorphism defined by

$$\sigma_a(z) = \frac{a-z}{1-\bar{a}z}, \quad z \in \mathbb{D}.$$

A key tool for our results is the submean value property for the generalized counting function $N_{\varphi,\alpha}$. This fact is deduced from the corresponding property for the classical Nevanlinna counting function together with the following formula due to A. Aleman (see Lemma 2.3 in [4]) that relates the function $N_{\varphi,\alpha}$ with the classical Nevanlinna counting function N_φ.

Aleman's formula. Let $0 < \alpha < 1$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic and nonconstant. Then for every $\zeta \in \varphi(\mathbb{D})$,

$$N_{\varphi,\alpha}(\zeta) = -\frac{1}{2} \int_{\mathbb{D}} \Delta\omega_\alpha(z)\, N_{\varphi\circ\sigma_z}(\zeta)\, dA(z), \tag{2.1}$$

where $\omega_\alpha(z) = (1-|z|^2)^\alpha$, and Δ denotes the standard Laplace operator.

Proposition 2.1. *Let $0 < \alpha < 1$ and $0 < p < \infty$. Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic. Then there is a positive constant $C = C_p < \infty$ such that*

$$N_{\varphi,\alpha}(\zeta)^p \le \frac{C}{|B|} \int_B N_{\varphi,\alpha}(w)^p\, dA(w),$$

where $\zeta \in \mathbb{D} \setminus \{\varphi(0)\}$ and B is any euclidean disk centered at ζ contained in $\mathbb{D} \setminus \{\varphi(0)\}$. Moreover, one can take $C = 1$ if $p \ge 1$.

Proof. We begin with the case $p = 1$. This case has been proved recently in [10]. Let $\omega_\alpha(z) = (1-|z|^2)^\alpha$. An application of Aleman's formula together with the submean value property for the classical Nevanlinna counting function, and taking into account that $\Delta\omega_\alpha(z) \le 0$, gives

$$N_{\varphi,\alpha}(\zeta) \le -\frac{1}{2} \int_{\mathbb{D}} \Delta\omega_\alpha(z) \left(\frac{1}{|B|} \int_B N_{\varphi\circ\sigma_z}(w)\, dA(w) \right) dA(z),$$

for any euclidean disk B centered at ζ contained in $\mathbb{D} \setminus \{\varphi(0)\}$. Now, we continue by using Fubini's theorem together with another application of Aleman's formula to obtain

$$\begin{aligned} N_{\varphi,\alpha}(\zeta) &\le \frac{1}{|B|} \int_B \left(-\frac{1}{2} \int_{\mathbb{D}} \Delta\omega_\alpha(z)\, N_{\varphi\circ\sigma_z}(w)\, dA(z) \right) dA(w) \\ &= \frac{1}{|B|} \int_B N_{\varphi,\alpha}(w)\, dA(w), \end{aligned}$$

which proves the desired result for $p = 1$ with $C = 1$. This special case together with Hölder's inequality then implies that the desired inequality holds with $C = 1$ for all $p \ge 1$.