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On the full Kostant–Toda system and the discrete Korteweg–de Vries equations

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ABSTRACT

The relation between the solutions of the full Kostant–Toda lattice and the discrete Korteweg–de Vries equation is analyzed. A method for constructing solutions of these systems is given. As a consequence of the matricial interpretation of this method, the transform of Darboux is extended for general Hessenberg banded matrices.

1. Introduction

In [10,7] the construction of a solution of the Toda lattice

$$\left.\begin{aligned} \dot{a}_n &= b_n - b_{n-1} \\ \dot{b}_n &= b_n(a_{n+1} - a_n) \end{aligned}\right\}, \quad n \in \mathbb{N}, \tag{1}$$

from another given solution was studied. Both solutions of (1) were linked to each other by a Backlünd transformation, also called the Miura transformation, given by

$$\left.\begin{aligned} b_n &= \gamma_{2n}\gamma_{2n+1}, \qquad a_n = \gamma_{2n-1} + \gamma_{2n} + C \\ \tilde{b}_n &= \gamma_{2n+1}\gamma_{2n+2}, \qquad \tilde{a}_n = \gamma_{2n} + \gamma_{2n+1} + C \end{aligned}\right\}$$

where $\{\gamma_n\}$ is a solution of the Volterra lattice

$$\dot{\gamma}_n = \gamma_n \left(\gamma_{n+1} - \gamma_{n-1}\right). \tag{2}$$

Here and in the following, the dot means differentiation with respect to $t \in \mathbb{R}$. However, we suppress the explicit t-dependence for brevity.

In [2], the first and the second authors generalized the analysis given in [7] to the kind of Toda and Volterra lattices studied in [1]. As a particular case, the results obtained in [2] extend the corresponding of [7] to the case of Toda lattices

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where $a_n(t)$ and $b_n(t)$ are complex functions of $t \in \mathbb{R}$. Now, our goal is to extend the results of [2] and [7] to the complex full Kostant–Toda lattice, which is given by

$$\left.\begin{array}{ll} \dot{a}_n^{(1)} & = a_n^{(2)} - a_{n-1}^{(2)} \\ \dot{a}_n^{(2)} & = \left(a_{n+1}^{(1)} - a_n^{(1)}\right) a_n^{(2)} + a_n^{(3)} - a_{n-1}^{(3)} \\ \dot{a}_n^{(3)} & = \left(a_{n+2}^{(1)} - a_n^{(1)}\right) a_n^{(3)} + a_n^{(4)} - a_{n-1}^{(4)} \\ \quad\vdots & \\ \dot{a}_n^{(p-1)} & = \left(a_{n+p-2}^{(1)} - a_n^{(1)}\right) a_n^{(p-1)} + a_n^{(p)} - a_{n-1}^{(p)} \\ \dot{a}_n^{(p)} & = \left(a_{n+p-1}^{(1)} - a_n^{(1)}\right) a_n^{(p)} \end{array}\right\}, \quad n \in \mathbb{N}. \tag{3}$$

In the sequel, for each $n \in \mathbb{N}$ we assume that $a_n^{(i)}$, $i = 1, 2, \ldots, p$, in (3) are continuous functions with complex values defined in the open interval $\mathcal{I}_n$ such that

$$\bigcap_{n=1}^{N} \mathcal{I}_n \neq \emptyset, \quad \text{for all } N \in \mathbb{N}. \tag{4}$$

It is easy to check that these equations can be formally written in a Lax pair form $\dot{J} = [J, J_-]$, where $[M, N] = MN - NM$ is the commutator of the operators M and N, and J, J_- are the operators whose matricial representation is given, respectively, by the banded matrices

$$J = \begin{pmatrix} a_1^{(1)} & 1 & & & & \\ a_1^{(2)} & a_2^{(1)} & 1 & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ a_1^{(p)} & a_2^{(p-1)} & \cdots & a_p^{(1)} & 1 & \\ & \ddots & \ddots & & \ddots & \ddots \end{pmatrix}, \qquad J_- = \begin{pmatrix} 0 & & & & \\ a_1^{(2)} & 0 & & & \\ \vdots & \vdots & \ddots & & \\ a_1^{(p)} & a_2^{(p-1)} & \cdots & 0 & \\ & \ddots & \ddots & & \ddots \end{pmatrix}, \tag{5}$$

and where J_- is the lower triangular part of J.

In this paper we do not distinguish between each operator and its matricial representation. Moreover, we underline the formal sense of the Lax pair expression. In fact, it could be that there is no open interval where all the entries of J are defined.

Definition 1. We say that J is a *solution* of (3) if its entries verify (3)–(4).

An important tool in the study of these systems is the sequence of polynomials $P_n(z) = P_n(t, z)$ associated with the matrix J, i.e., the polynomials defined by the following recurrence relation, given for $n = 0, 1, \ldots$ by

$$\left.\begin{array}{l} \displaystyle\sum_{i=1}^{p-1} a_{n-p+i+1}^{(p-i+1)} P_{n-p+i}(z) + (a_{n+1}^{(1)} - z) P_n(z) + P_{n+1}(z) = 0 \\ P_0(z) = 1, \qquad P_{-1}(z) = \cdots = P_{-p+1}(z) = 0. \end{array}\right\} \tag{6}$$

We will use the following well-known fact:

$$P_n(z) = \det\left(zI_n - J_n\right), \quad n \in \mathbb{N}. \tag{7}$$

Here and in the sequel we denote by A_n the finite matrix formed with the first n rows and n columns of each infinite matrix A.

For each $p \in \mathbb{N}$, the system

$$\dot{\gamma}_n = \gamma_n \left(\sum_{i=1}^{p-1} \gamma_{n+i} - \sum_{i=1}^{p-1} \gamma_{n-i} \right), \quad n \in \mathbb{N}, \tag{8}$$

called discrete Korteweg–de Vries equations, is an extension of the Volterra lattices (2) studied in [10,7] (see also [6]). As in (3)–(4), for each $n \in \mathbb{N}$ we assume that the functions in (8) are continuous and defined in an open set $\mathcal{O}_n$ such that

$$\bigcap_{n=1}^{N} \mathcal{O}_n \neq \emptyset, \quad \text{for all } N \in \mathbb{N}. \tag{9}$$