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Bifurcation curves of a diffusive logistic equation with harvesting orthogonal to the first eigenfunction

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ABSTRACT

We study the global bifurcation curves of a diffusive logistic equation, when harvesting is orthogonal to the first eigenfunction of the Laplacian, for values of the linear growth up to $\lambda_2 + \delta$, examining in detail their behavior as the linear growth rate crosses the first two eigenvalues. We observe some new behavior with regard to earlier works concerning this equation. Namely, the bifurcation curves suffer a transformation at λ_1, they are compact above λ_1, there are precisely two families of degenerate solutions with Morse index equal to zero, and the whole set of solutions below λ_2 is not a two dimensional manifold.

1. Introduction

This paper concerns the study of logistic equations of the form

$$-\Delta u = au - f(u) - ch, \tag{1}$$

in a smooth bounded domain $\Omega \in \mathbb{R}^N$, with $N \geq 1$. We are interested in weak solutions belonging to the space

$$\mathcal{H} = \{u \in W^{2,p}(\Omega) : u = 0 \text{ on } \partial\Omega\},$$

for some fixed $p > N$. Let λ_1 and λ_2 be the first and second eigenvalues of the Dirichlet Laplacian on Ω, respectively. We denote by ϕ the first eigenfunction satisfying $\max_\Omega \phi = 1$. We assume that λ_2 is simple, with eigenspace spanned by ψ, and we also normalize the second eigenfunction so $\max_\Omega \psi = 1$.

The competition term f is assumed to satisfy the following hypotheses:

(i) $f \in C^2(\mathbb{R})$.
(ii) $f(u) = 0$ for $u \leq M$, and $f(u) > 0$ for $u > M$; throughout $M \geq 0$ is fixed.
(iii) $f''(u) \geq 0$.
(iv) $\lim_{u\to+\infty} \frac{f(u)}{u} = +\infty$.

In [13], the authors obtained global bifurcation curves, of positive solutions to (1), for values of the parameter a in a right neighborhood of λ_1, when $f(u) = u^2$ and h is a positive function.

In [10], the first author generalized the results of [13] to competition terms satisfying (i)–(iv), and studied the bifurcation curves, of sign changing solutions, for a up to $\lambda_2 + \delta$, for some $\delta > 0$. This was also done under the assumption that h was positive a.e. in Ω, a hypothesis which was used in the proof, although, as noted in [10], in a right neighborhood of λ_1, one may relax the requirement on h to $\int_\Omega h\phi\, dx \neq 0$.

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In this paper, we analyze the situation when the harvesting function h, which in our case might be more appropriately called harvesting and plantation function, is orthogonal to the first eigenfunction of the Laplacian. The biological interpretation gives our context but should be taken with care, as it breaks down in several circumstances, for instance, if the solutions become negative. Our motivation is mathematical, we are forced to provide new arguments, and we suspected the geometry of the problem would be different from the one in [10]. Indeed, it turns out that the bifurcation curves suffer a complete transformation when the parameter a crosses the first eigenvalue. We examine in detail the way in which this change occurs. When seen in the (a, u, c) space, the set of solutions of (1) between λ_1 and λ_2 has the shape of a piece of a paraboloid, with a flat bottom at $a = \lambda_1$. A 2-dimensional space of solutions is attached to this bottom at $a = \lambda_1$, along a segment, and lies in the region $a \le \lambda_1$. The whole set of solutions below λ_2 is not a two-dimensional manifold. Therefore we find a richer behavior regarding this equation than in the earlier works. Also, in contrast to the bifurcation curves obtained in the previous papers, our curves turn out to be compact above λ_1, and, instead of one, we get two families of degenerate solutions with Morse index equal to zero above λ_1.

Specifically, we assume:

(a) $h \in L^\infty(\Omega)$.
(b) $\int_\Omega h\phi\, dx = 0$.
(c) $\int_\Omega h\psi\, dx \neq 0$.

Hypothesis (c) also appears in [10]. Our main results are Theorems 3.2, 4.8, 4.11, 4.12, 5.1 and 5.2. The proofs involve bifurcation methods [7,8], a blow up argument, the Morse indices, and a careful choice of coordinates at each step. In particular, around λ_1 we decompose the space $\mathcal{H}$ as in [3]. In the end, we obtain a complete picture of the set of solutions for the parameter a up to $\lambda_2 + \delta$.

For other works related to logistic equations with harvesting we refer the reader to [5,11,14].

This paper is organized as follows: We treat successively the cases where the linear growth parameter a is equal to λ_1 (Section 2), below λ_1 (Section 3), between λ_1 and λ_2 (Section 4), and greater than or equal to λ_2 (Section 5).

2. Linear growth a equal to λ_1

Assume $(\lambda_1, u, c) \in \mathbb{R} \times \mathcal{H} \times \mathbb{R}$ is a solution of (1) with $a = \lambda_1$. Multiplying both sides of (1) by ϕ and integrating, taking into account (b) and $-\Delta\phi = \lambda_1\phi$, we deduce that $\int f(u)\phi\, dx = 0$. When the region of integration is omitted it is understood to be Ω. Because ϕ is positive in Ω and $f(u)$ is continuous in Ω, we get that $f(u) \equiv 0$. This means $u \le M$ by (ii). Therefore, for $a = \lambda_1$ the solutions of (1) are those of the linear problem

$$-\Delta u = \lambda_1 u - ch,$$

i.e. are of the form (λ_1, u, c), where $u = t\phi + c(\Delta + \lambda_1)^{-1}h$, with

$$(t, c) \in \Lambda := \{(t, c) \in \mathbb{R}^2 : t\phi + c(\Delta + \lambda_1)^{-1}h \le M\}. \tag{2}$$

Thus, there is a bijection between the set of solutions of (1) for $a = \lambda_1$ and Λ, given by $(\lambda_1, t\phi + c(\Delta + \lambda_1)^{-1}h, c) \leftrightarrow (t, c)$. The set Λ is closed and convex.

Let

$$T := \sup\{t : \text{there exists } c \text{ such that } (t, c) \in \Lambda\}. \tag{3}$$

Taking $c = 0$ and using the normalization $\max_\Omega \phi = 1$, $(M, 0) \in \Lambda$ and so $T \ge M$. The value of T is finite. Indeed, h and $(\Delta + \lambda_1)^{-1}h$ are orthogonal to ϕ and so $(\Delta + \lambda_1)^{-1}h$ changes sign. Let Ω_+ be the set where $(\Delta + \lambda_1)^{-1}h$ is positive and let Ω_- be the set where $(\Delta + \lambda_1)^{-1}h$ is negative. Suppose that $c \ge 0$; then $M - c(\Delta + \lambda_1)^{-1}h \le M$ on Ω_+, and so if $(t, c) \in \Lambda$, then $t\phi \le M$ on Ω_+. Suppose $c < 0$; then $M - c(\Delta + \lambda_1)^{-1}h \le M$ on Ω_-, and so if $(t, c) \in \Lambda$, then $t\phi \le M$ on Ω_-. In any case, $c \ge 0$ or $c < 0$, $t\phi \le M$ either on Ω_+ or on Ω_-. We conclude that $T < +\infty$ as asserted. The value T is a maximum.

To characterize parts of the boundary of Λ, we define two functions, $c^-_{\lambda_1}$ and $c^+_{\lambda_1}$, in the interval $]-\infty, T]$, by

$$c^-_{\lambda_1}(t) := \min_{(t,c)\in\Lambda} c \quad \text{and} \quad c^+_{\lambda_1}(t) := \max_{(t,c)\in\Lambda} c. \tag{4}$$

Clearly, $c^-_{\lambda_1}(T) \le c^+_{\lambda_1}(T)$. Notice that

$$\lim_{t\to-\infty} c^-_{\lambda_1}(t) = -\infty \quad \text{and} \quad \lim_{t\to-\infty} c^+_{\lambda_1}(t) = +\infty, \tag{5}$$

since, when $t \to -\infty$, denoting by ν the unit outward normal to Ω, using Hopf's Lemma, we have $\frac{\partial}{\partial\nu}(M - t\phi) = -t\frac{\partial\phi}{\partial\nu} \ge -t \max_{\partial\Omega} \frac{\partial\phi}{\partial\nu} \to -\infty$, and $M - t\phi \to +\infty$ uniformly in each compact subset of Ω. Therefore, because $(\Delta + \lambda_1)^{-1}h$ belongs to $C^1(\overline{\Omega})$, as t goes to $-\infty$, it is possible to guarantee that $c(\Delta + \lambda_1)^{-1}h \le M - t\phi$ for larger and larger values of $|c|$. This establishes (5). From (5) and the fact that Λ is convex, it follows that $c^-_{\lambda_1}$ is convex, continuous and strictly increasing. Similarly, $c^+_{\lambda_1}$ is concave, continuous and strictly decreasing.