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CEV asymptotics of American options

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ABSTRACT

The constant elasticity of variance (CEV) model is a practical approach to option pricing by fitting to the implied volatility smile. Its application to American-style derivatives, however, poses analytical and numerical challenges. By taking the Laplace–Carson transform (LCT) to the free-boundary value problem characterizing the option value function and the early exercise boundary, the analytical result involves confluent hyper-geometric functions. Thus, the numerical computation could be unstable and inefficient for certain set of parameter values. We solve this problem by an asymptotic approach to the American option pricing problem under the CEV model. We demonstrate the use of the proposed approach using perpetual and finite-time American puts.

1. Introduction

American options are options that can be exercised at any time prior to maturity. The early exercise feature makes American options more valuable and attractive in the financial market. However, the optimal exercise boundary (strategy) also makes the valuation of American options a highly challenging problem in finance because the corresponding valuation is related to an optimal stopping problem in probability or a free-boundary value problem in partial differential equations (PDEs). Apart from American calls and puts, many financial products essentially belong to the class of American options. For instance, a stock loan can be transformed into a perpetual American call option [13]. Russian option or no-regret option is the nickname of the perpetual American lookback option [7]. Therefore, the valuation of American options is the central problem in this paper.

Empirical studies suggest that the Black–Scholes (BS) model is inadequate to explain the volatility smile observed in the financial market. One popular alternative is the constant elasticity of variance (CEV) model, introduced by Cox [3], which captures the volatility smile through a power of the underlying asset price as the local volatility of the asset price process. The CEV model has stimulated many interesting studies since [3]. For instance, Emanuel and MacBeth [5] derive a closed-form solution for European options under the CEV model. Davydov and Linetsky [4] apply the Laplace transform to obtain analytical formulas for the prices of several path-dependent options under the CEV model. Wong and Zhao [11] construct an artificial boundary finite difference method to compute American options under the CEV model. By means of the homotopy analysis method, Zhao and Wong [15] derive a closed-form solution for American option prices under general diffusion which nests the CEV model. However, the implementation of the homotopy solution requires a complicated iterative integration. Wong and Zhao [12] derive closed-form solutions for American put under the CEV model by taking the Laplace–Carson transform (LCT).

Although there are several analytical results on American option pricing under CEV in the literature, their numerical use is far from being satisfactory. Specifically, the pricing formula in [12] requires the computation of confluent hyper-geometric functions followed by a Laplace inversion. The numerical implementation is hardly made in a stable and efficient manner.

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For instance, we experience that their analytical formula fails to produce a numerical output when the elasticity of variance is negative and close to zero. This motivates us to apply the perturbation technique to the free-boundary value problems of American options with respect to the elasticity of variance.

Our approach is based on Park and Kim [8], who propose an asymptotic PDE approach to stabilize the numerical computation of path-dependent options under CEV. An asymptotic analysis of a similar model but in a different limit is also addressed in [14]. Similar asymptotic PDE approach also appears in the literature of stochastic volatility model such as [2] and reference therein. However, these studies only consider European-style contract without taking into account the optimal exercise decision.

Recently, Wong and Wong [10] have constructed an asymptotic approach for stock loan under the fast mean-reverting stochastic volatility model. As the valuation of stock loan involves the optimal exercise boundary, their asymptotic analysis contains two asymptotic expansions for the value function and the optimal exercise boundary, respectively. These two expansion formulas are simultaneously solved from the free-boundary value problem governing the stock loan. Inspired by them, this paper combines the approaches of [8,10] to American option pricing problems under CEV. We investigate both perpetual and finite-time American options.

For finite maturity American options, we first apply the LCT to the option value function and the early exercise boundary as proposed by [1,12]. The problem is then transformed to the one related to perpetual American options. Under the assumption of a small value of elasticity of variance, we derive the analytical recursive formulas for each order of approximation in the asymptotic expansion. We show numerically that the asymptotic formulas are well behaved and offer reasonable numerical values when traditional analytical techniques fail to do so.

The remainder of this paper is organized as follows. Section 2 introduces the CEV model and the free-boundary value problem for American options. Section 3 presents the asymptotic solutions of American option price and its optimal exercise boundary. Section 4 provides numerical examples to illustrate the asymptotic solutions. Section 5 concludes this paper and suggests possible future works.

2. Problem formulation

2.1. The CEV model

The CEV model is defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where the probability measure $\mathbb{P}$ is the market-implied risk-neutral probability, and the filtration $\mathcal{F}$ is the σ-field generated by the risk-neutral process $\{S_s\}_{0\leq s\leq t}$ that satisfies the stochastic differential equation (SDE): $\frac{dS_t}{S_t} = (r-q)\,dt + \delta S_t^{\beta}\,dW_t$, where r is the risk-free interest rate, q is the dividend yield, and W_t is the Wiener process. The parameter β is known as the elasticity of variance ($\frac{S}{\sigma}\frac{d\sigma}{dS} = \beta$) while δ is the scale parameter fixing the initial instantaneous volatility ($\sigma_0 = \delta S_0^{\beta}$), where the local volatility σ is the diffusion coefficient.

As stated in [4], the CEV model nests several financial models as its special cases, such as the BS model and the square-root volatility model. The value of β controls the slope of the volatility smile implied by European options. Empirical studies suggest that β is usually negative and close to zero for the fact that implied volatility smiles are usually downward sloping. When $\beta = 0$, the CEV model reduces to the BS model. Hence, our asymptotic expansion is perturbed around the BS solution.

The put-call symmetry of American options enables the American call pricing formula to be inferred by its put counterpart. Thus, we focus on American put in this paper. However, the proposed approach is generally useful for a wide range of American options on single asset.

2.2. Perpetual American put

To simplify matters, we begin with the perpetual American put. The no-arbitrage price of this option is given as follows.

$$V(S) = \operatorname*{ess\,sup}_{\theta\in\mathcal{T}_\infty} \mathbb{E}[e^{-r\theta}(K-S_\theta)^+ | S_t = S] \tag{1}$$

where $(\bullet)^+ = \max(\bullet, 0)$, and $\mathcal{T}_\infty$ is the set of stopping times. Specifically, $\mathcal{T}_\infty = \inf\{\xi > t | V(S_\xi) \leq K - S_\xi\}$, which can be shown to be equivalent to $\mathcal{T}_\infty = \inf\{\xi > t | S_\xi \leq S^*\}$, where the constant, S^*, is the optimal exercise boundary that maximizes the value function V. Note that all expectations in this paper are taken under the risk-neutral probability $\mathbb{P}$.

It is well known that the problem in (1) can be formulated as a free-boundary value problem [6] as follows.

$$\frac{1}{2}\delta^2 S^{2\beta+2}\frac{d^2V}{dS^2} + (r-q)S\frac{dV}{dS} - rV = 0, \quad S \geq S^*, \tag{2}$$

$$V(S^*) = K - S^*, \tag{3}$$

$$\frac{dV(S^*)}{dS} = -1, \tag{4}$$

$$\lim_{S\to\infty} V(S) = 0, \tag{5}$$