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Bounds on Kuhfittig's iteration schema in uniformly convex hyperbolic spaces

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ABSTRACT

The purpose of this paper is to extract an explicit effective and uniform bound on the rate of asymptotic regularity of an iteration schema involving a finite family of nonexpansive mappings. The results presented in this paper contribute to the general project of proof mining as developed by the second author as well as generalize and improve various classical and corresponding quantitative results in the current literature. More precisely, we give a rate of asymptotic regularity of an iteration schema due to Kuhfittig for finitely many nonexpansive mappings in the context of uniformly convex hyperbolic spaces. The rate only depends on an upper bound on the distance between the starting point and some common fixed point, a lower bound $1/N \leq \lambda_n(1-\lambda_n)$, the error $\epsilon > 0$ and a modulus η of uniform convexity.

1. Introduction and preliminaries

This paper is a continuation of the case study in the general program of 'proof mining' – introduced by Kohlenbach in the 90s (see [13]) – which provides proof-theoretic tools to extract explicit and effective uniform bounds from ineffective proofs in functional analysis and in particular metric fixed point theory and ergodic theory. For various classes of proofs, the extraction of such effective uniform bounds is guaranteed by so-called 'logical metatheorems'. These metatheorems were developed in [14] for abstract bounded metric structures (including hyperbolic and $CAT(0)$ spaces) and bounded convex subsets of a normed space and generalized to unbounded structures in [7]. In [26] this was adapted to further structures such as $\mathbb{R}$-trees, Gromov's δ-hyperbolic spaces and uniformly convex W-hyperbolic spaces. In the context of these spaces, the metatheorems cover functions such as nonexpansive, Lipschitz, weakly quasi-nonexpansive or uniformly continuous maps among others. As an application of these metatheorems, strong uniform bounds have been extracted from numerous previously established convergence results in metric fixed point and ergodic theory, see, for example, [15–24,27,28,31] and the references cited therein.

Fixed point theory for nonexpansive mappings is an active area of research in nonlinear functional analysis and it requires tools far beyond from metric fixed point theory. The problem of finding a common fixed point of a finite family of nonlinear mappings acting on a nonempty convex domain often arises in applied mathematics. For example, finding a common fixed point of a finite family of nonexpansive mappings may be used to solve a convex minimization problem or a system of simultaneous equations. Hence, the analysis of iterative schemas for finite families of nonexpansive mappings is a problem of interest in such contexts.

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In fixed point theory, various iterative schemas $\{x_n\}$ for computing fixed points of nonlinear mappings T have been studied. One of the most important notions in metric fixed point theory is the asymptotic regularity [5] of the iteration under consideration, i.e.

$$d(x_n, T(x_n)) \stackrel{n\to\infty}{\to} 0. \tag{$*$}$$

This usually is an important first step in establishing (under suitable additional assumptions such as compactness conditions) the convergence of $\{x_n\}$ towards a fixed point. The concept of asymptotic regularity naturally leads to the issue of finding a rate of convergence in $(*)$ or even for the convergence of $\{x_n\}$ itself (in cases where strong convergence holds). Whereas the latter usually can be shown to be inherently noncomputable and to be effectively solvable only in the computationally weaker (though ineffectively equivalent) formulation of metastability (in the sense of Tao [33])

$$\forall \epsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n\, \forall i, j \in [n, n + g(n)] \quad (d(x_i, x_j) < \epsilon),$$

rates of asymptotic regularity typically can be obtained. A series of papers with case studies in the general program of 'proof mining' has been carried out for the extraction of uniform bounds by examining the Krasnoselskii–Mann iteration of nonexpansive mappings. In [15], the second author analyzed a result due to Borwein–Reich–Shafrir [2] and obtained effective uniform bounds on the asymptotic behavior of the Krasnoselskii–Mann iteration which was extended in [20] to W-hyperbolic spaces and directionally nonexpansive mappings. Later on, in 2003, simpler bounds on asymptotic regularity of the Krasnoselskii–Mann iteration were computed in uniformly convex normed spaces [16] for more general iterations considered by Groetsch [10]. It is worth mentioning that the bounds on the rate of asymptotic regularity established in [15, 16] (see also, [1,12]) are related to single nonexpansive mappings. It is natural to analyze also proofs of convergence results for families of nonexpansive mappings for the extraction of bounds on asymptotic regularity. So far, no such bounds have been extracted in this context.

In 1981, Kuhfittig [25] established a fundamental theorem regarding the approximation of a common fixed point of a finite family of nonexpansive mappings in a strictly convex Banach space. More precisely, he proposed the following iteration schema:

Let C be a nonempty convex compact subset of a Banach space E and $\{T_i : 1 \le i \le k\}$ be a finite family of nonexpansive self-mappings with a nonempty set of common fixed points $F = \cap_{i=1}^{k} F(T_i) \neq \emptyset$. Let $U_0 = I$ be the identity mapping and $0 < \lambda < 1$, then using the mappings

$$\begin{aligned} U_1 &= (1-\lambda)I + \lambda T_1 U_0 \\ U_2 &= (1-\lambda)I + \lambda T_2 U_1 \\ &\vdots \\ U_k &= (1-\lambda)I + \lambda T_k U_{k-1}, \end{aligned}$$

one defines

$$x_0 \in C, \qquad x_{n+1} := (1-\lambda)x_n + \lambda T_k U_{k-1} x_n, \quad n \ge 0. \tag{1.1}$$

If $k = 1$, then the iteration schema (1.1) reduces to the usual Krasnoselskii–Mann iteration of T_1 (for constant λ)

$$x_{n+1} = (1-\lambda)x_n + \lambda T_1 x_n,$$

which contains the Krasnoselskii iteration as a special case for $\lambda = \frac{1}{2}$. The strong convergence of $\{x_n\}$ in strictly convex Banach spaces is established in [25] under the assumption of the compactness of C. The weak convergence in uniformly convex Banach spaces satisfying Opial's is proved under the assumption that C is closed and convex. In 2000, Rhoades [30] managed to eliminate the assumption of the Opial condition from the latter result.

Implicit in Kuhfittig's paper [25] is the following asymptotic regularity result:

Theorem 1.1. *Let E be a strictly convex Banach space, C a nonempty compact convex subset of E and $T_1, T_2, \ldots, T_k$ a family of nonexpansive self-mappings of C with $F \neq \emptyset$. Let $x_0 \in C$, then for the sequence $\{x_n\}$ generated by (1.1), we have*

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0 \quad \textit{for all } 1 \le i \le k.$$

Kuhfittig argues using the functions $S_i := T_i U_{i-1}$ for $1 \le i \le k$ which again are nonexpansive. Since $\{x_n\}$ is a Krasnoselskii–Mann iteration of S_k it follows by classical theorems due to Edelstein and Ishikawa that $\{x_n\}$ converges strongly to a fixed point p of S_k. He then shows

$$\forall q \in C \quad \left(S_k(q) = q \Longrightarrow \bigwedge_{i=1}^{k} (S_i q = q) \Longrightarrow \bigwedge_{i=1}^{k} (T_i q = q) \right).$$

Taking than $q := p$ gives the strong convergence of $\{x_n\}$ to a common fixed point of $T_1, \ldots, T_k$ and so a fortiori Theorem 1.1. That latter theorem, however, already follows without any compactness assumption on C provided that E is uniformly convex.