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Symmetry and uniqueness of nonnegative solutions of some problems in the halfspace

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ABSTRACT

We derive some 1-D symmetry and uniqueness or non-existence results for nonnegative solutions of

$$\begin{cases} -\mathrm{div}\,(A(x)\nabla u) = u - g(x) & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+ \end{cases}$$

in low dimension, under suitable assumptions on A and g. Our method is based upon a combination of Fourier series and Liouville theorems.

1. Introduction

This paper concerns symmetry and uniqueness or non-existence of nonnegative solutions to problems of type

$$\begin{cases} -\mathrm{div}\,(A(x)\nabla u) = u - g(x) & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{1}$$

in low dimension; here $\mathrm{div}\,(A(x)\nabla)$ is an elliptic operator (not necessarily uniformly elliptic). As far as g is concerned, we will see that the existence and the properties of nonnegative solutions to (1) depend strongly on it. With this in mind, we will start considering as model problem

$$\begin{cases} -\Delta u = u - 1 & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{2}$$

In Section 2, we will prove the following statement.

Theorem 1.1. *Let $N = 2$ or 3. If $u \in \mathcal{C}^2(\overline{\mathbb{R}}^N_+)$ solves problem (2) and*

$$\forall M > 0 \quad \exists C(M) > 0 : 0 \le u(x) \le C(M) \; \forall x \in \mathbb{R}^{N-1} \times [0, M], \tag{3}$$

then

$$u(x', x_N) = 1 - \cos x_N.$$

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This is a result of uniqueness and of 1-D symmetry, i.e. the (unique) solution of (2) is a function depending only on x_N. Note that assumption (3) means that u is nonnegative and bounded in every strip of type $\mathbb{R}^{N-1} \times [0, M]$.

Unfortunately, we will see that the assumption "$N = 2$ or 3" is substantial for our proof. However, we can still say something for the model problem in higher dimension. This will be the object of Section 2.1.

As far as the generalization towards problem (1) is concerned, we will see in Section 3, Theorem 3.2, that the presence of $\operatorname{div}(A(x)\nabla)$ instead of the Laplacian does not affect the previous result, under suitable assumptions on A.

A further natural generalization of problem (2) consists in introducing a g depending only on x_N instead of the constant function 1:

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = u - g(x_N) & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{4}$$

In this setting, Theorem 4.7 is the counterpart of Theorem 1.1; as an immediate corollary we have

Corollary 1.2. *Let $N = 2$ or 3. Under suitable assumptions on A and on g, if $u \in \mathcal{C}^2(\overline{\mathbb{R}}^N_+)$ solves (4) and satisfies (3), then u is uniquely determined and depends only on x_N.*

Finally, always in Section 4, we will show how to use the method developed in the previous sections in order to deal with a wider class of inhomogeneous terms (depending also on x'), obtaining sharp results for some particular cases; for instance, we will see that if $g = g(x')$ and there exists a solution u of (1) satisfying (3), then g has to be constant.

The interest in the model problem comes from Berestycki, Caffarelli and Nirenberg: in [2] they proved that a *positive and bounded* solution to (2) does not exist when $N \leq 3$. Their result fits in a wider study of 1-D symmetry and monotonicity for *positive and bounded* solutions to

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{5}$$

with f Lipschitz continuous. If $N \geq 2$ and $f(0) \geq 0$, then a positive and bounded solution is strictly increasing in the x_N variable (see [2,3]). Furthermore, always in [2], if $N \leq 3$, $f \in \mathcal{C}^1(\mathbb{R})$ and $f(0) \geq 0$, they showed that a positive and bounded solution depends only on one variable (1-D symmetry). Another contribution, contained in [2], is that the monotonicity and the 1-D symmetry hold true for $N = 2$ without any restriction on the sign of $f(0)$. The proofs of the quoted results are based on the moving planes method and on a previous result in [1], where it is shown that if u is a positive and bounded solution of (5) and

$$f(M) \leq 0 \quad \text{where } M = \sup_{x \in \mathbb{R}^N_+} u(x),$$

then u is symmetric and monotone, and $f(M) = 0$. When f is a power (thus $f(0) = 0$), similar results has been achieved in [7,8]. We point out that our contribution is not included in the existing literature, because we are considering *nonnegative and not necessarily bounded* solutions, and because in general we are interested in the case $f(0) < 0$. In such a situation the moving planes method gives just partial results, as shown by Dancer [4]. We emphasize the fact that the difference between *positive* and *nonnegative* is substantial for $f(0) < 0$, since in this case natural solutions are nonnegative and non-monotone, and a positive solution does not necessarily exists; this is clearly the case of the model problem (2). For all these reasons, our approach is different and it is based upon a combination of Fourier series and Liouville theorems.

To complete the essential bibliography for this kind of problems, we mention also the work [6], where symmetry and monotonicity are obtained under weaker regularity assumptions on f, and an extension in dimension 4 and 5 is given for a wide class of nonlinearities.

Notation. We will consider problems in the half space $\mathbb{R}^N_+ := \mathbb{R}^{N-1} \times (0, +\infty)$. As usual, we will denote by (x', x_N) a point of $\mathbb{R}^N_+$.

The symbols ∇', div' or Δ' will be used respectively for the gradient, the divergence or the Laplacian in $\mathbb{R}^{N-1}$.

The notation u_j will be used to indicate the partial derivative of u with respect to the x_j variable. For any $x \in \mathbb{R}^N$, for any $R > 0$, we will write $B_R(x)$ to indicate the ball of centre x and radius R. If $x = 0$, we will simply write B_R.

For any $A \subset \mathbb{R}^N$, χ_A will denote the characteristic function of A.

We will use the notation $\langle \cdot, \cdot \rangle$ for the usual scalar product in any euclidean space.

Given a real valued function v, we denote its positive part as v^+.

2. The model problem

In this section we consider problem (2):

$$\begin{cases} -\Delta u = u - 1 & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases}$$

We aim at proving Theorem 1.1.