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Mixed A_1–A_∞ bounds for fractional integrals

Jorgelina Recchi

Departamento de Matemática, Universidad Nacional del Sur, 8000 Bahía Blanca, Argentina

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ABSTRACT

For a fractional integral operator I_α we show that

$$\|I_\alpha f\|_{L^q(w^q)} \leq c[w]_{A_{\infty,q}}^{1/p'}[w]_{A_{1,q}}^{1/q}\|f\|_{L^p(w^p)}$$

where

$$[w]_{A_{\infty,q}} \equiv \sup_Q \frac{1}{w^q(Q)}\int_Q M(\chi_Q w^q)dx$$

is Wilson $A_{\infty,q}$'s constant which is smaller than the $A_{1,q}$ constant.

We also prove a weak type (1, 1) bound for a couple of weights of the form $(w, M_\alpha w)$ assuming that $w \in A_\infty$.

1. Introduction

The purpose of this paper is to study certain weighted estimates related to the fractional integral operator or Riesz potential I_α, defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}dy$$

and to the fractional maximal operator

$$M_\alpha f(x) = \sup_{Q\ni x} \frac{|Q|^{\alpha/n}}{|Q|}\int_Q |f(y)|dy$$

where $0 < \alpha < n$.

As usual, a non-negative locally integrable function, or weight, w is said to belong to the A_1 class if $Mw(x) \leq Cw(x)$, where M is the Hardy–Littlewood maximal operator

$$Mw(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q w(y)dy.$$

It is shown in [19] that if T is any Calderón–Zygmund operator and $1 < p < \infty$, then

$$\|T\|_{L^p(w)} \leq c_T\, pp'[w]_{A_1}. \tag{1.1}$$

This estimate improved the corresponding result previously obtained in [20] and is quite sharp since the following weighted endpoint estimate can be derived:

$$\|Tf\|_{L^{1,\infty}(w)} \leq c_{d,T}\,[w]_{A_1}\log(e+[w]_{A_1})\|f\|_{L^1(w)}. \tag{1.2}$$

E-mail addresses: drecchi@uns.edu.ar, jrecchi@gmail.com.

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More recently these two estimates have been improved in [13] by replacing a "fraction" of the A_1 constant by the A_∞ constant (see (1.11)). In fact, the following two estimates are shown in [13]:

$$\|T\|_{L^p(w)} \le c_T\, pp'\, [w]_{A_1}^{1/p}[w]_{A_\infty}^{1/p'}$$

and

$$\|Tf\|_{L^{1,\infty}(w)} \le c_T\, \log(e+[w]_{A_\infty})\|f\|_{L^1(Mw)}. \tag{1.3}$$

We remark that this estimate does not appear in [13] in this form. The one in [13] is a sharper version of (1.2), namely

$$\|Tf\|_{L^{1,\infty}(w)} \le c_T\, [w]_{A_1}\, \log(e+[w]_{A_\infty})\|f\|_{L^1(w)} \tag{1.4}$$

which follows from (1.3) by the definition of A_1. However, a simple modification of the proof of this inequality given in [13] yields easily (1.3).

One of the main results of this paper is to obtain similar inequalities for fractional integral operators.

Let w be a weight, $w \in A_{\infty,q}$ if

$$[w]_{A_{\infty,q}} \equiv \sup_Q \frac{1}{w^q(Q)} \int_Q M(\chi_Q w^q)dx < \infty.$$

Recall that $w \in A_{1,q}$ if there exists $c > 0$ such that

$$Mw^q \le c\ w^q \quad \text{a.e.}$$

and we denote by $[w]_{A_{1,q}}$ the smallest of these c's.

Theorem 1.1. *Let $1 < p < \frac{n}{\alpha}$, q be defined by the equation $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and $w \in A_{1,q}$. Then*

$$\|I_\alpha f\|_{L^q(w^q)} \le c[w]_{A_{\infty,q}}^{1/p'}[w]_{A_{1,q}}^{1/q}\|f\|_{L^p(w^p)}. \tag{1.5}$$

Furthermore, this inequality is sharp.

This result can be seen as a fractional version of Theorem 1.14 from [13]. However the reader should be warned that we have adopted a different notation here for the A_∞ constant. Also, it should be mentioned that another recent proof of (1.5) based on different techniques can be found in [7].

As a consequence of Theorem 1.1 we can deduce the following result.

Corollary 1.2. *Under the same condition as above we have*

$$\|I_\alpha f\|_{L^q(w^q)} \le c[w]_{A_{1,q}}^{1-\frac{\alpha}{n}}\|f\|_{L^p(w^p)} \tag{1.6}$$

where this estimate is sharp.

This corollary is the analogous estimate of (1.1) when T is a fractional integral operator and hence the L^p theory is clear. However, this is not the case when considering estimates like (1.2) or (1.3) when, again, T is replaced by a fractional integral. A closest antecedent can be found in [4] although one of the main results in this paper is negative. To be more precise, it is shown there that the following inequality

$$\|I_\alpha f\|_{L^{1,\infty}(w)} \le c\ \int_{\mathbb{R}^n} f(x)M_\alpha w(x)dx \tag{1.7}$$

does not hold in general. Now, if T is a Calderón–Zygmund operator and M is a Hardy–Littlewood maximal operator, the inequality analogous to (1.7) is the so called Muckenhoupt–Wheeden conjecture,

$$\sup_{t>0} t w(\{x \in \mathbb{R}^n : |Tf(x)| > t\}) \le c \int_{\mathbb{R}^n} |f|Mw(x)dx. \tag{1.8}$$

Also, a negative answer to (1.8) was provided by M. Reguera in [25] when T is a Haar multiplier and by M. Reguera and C. Thiele in [26] when T is the Hilbert transform.

In [4] the authors conjectured that the following estimate holds

$$\|I_\alpha f\|_{L^{1,\infty}(w)} \le c\ \int_{\mathbb{R}^n} f(x)M_\alpha Mw(x)dx. \tag{1.9}$$

If (1.9) were true, in particular, when $w \in A_1$ we would have

$$\|I_\alpha f\|_{L^{1,\infty}(w)} \le c\ [w]_{A_1}\int_{\mathbb{R}^n} f(x)M_\alpha w(x)dx.$$

Trying to give an answer to that conjecture, we obtained the following theorem.