J. Math. Anal. Appl. 402 (2013) 57–70

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

On the stability of m-fold circles and the dynamics of generalized curve shortening flows

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ARTICLE INFO

Article history:
Received 28 August 2012
Available online 11 January 2013
Submitted by Manuel del Pino

Keywords:
Curve-shortening
Symmetry
Blow-up
Exponential convergence

ABSTRACT

In this paper, we study the (asymptotic and exponential) stability of the m-fold circle as a solution of the p-curve shortening flow ($p \geq 1$ an integer).

1. Introduction

Let

$$x : \mathbb{S}^1 \times [0, T) \longrightarrow \mathbb{R}^2$$

be a family of smooth immersions of $\mathbb{S}^1$, the unit circle, into $\mathbb{R}^2$. We will say that x satisfies the p-curve shortening flow, $p \geq 1$, if x satisfies

$$\frac{\partial x}{\partial t} = -\frac{1}{p} k^p \mathbf{N}, \tag{1}$$

where k is the curvature of the embedding and $\mathbf{N}$ is the normal vector pointing outwards the region bounded by $x(\cdot, t)$.

Much is known about this family of flows. To select a few among many beautiful and fundamental works on the subject, we must mention the works of Gage and Hamilton [8], and Ben Andrews [2,3].

In this paper we will be concerned with the stability of m-fold circles as solutions to the p-curve shortening flow, p a positive integer. It is well known that there are certain small perturbations of the 2-fold circle such that the corresponding solution to the curve shortening flow does not behave asymptotically as a shrinking 2-fold circle, as it is described in [1] (see the discussion after Proposition 5.2 of that paper): the example basically consists on going around the circle twice, taking care that during the second turn we enlarge the circle a little bit, so the perturbed curve looks like a limacon; in this case, under the curve shortening flow, the inner loop will shrink faster than the outer loop and a cusp forms in finite time, and hence the curve does not shrink to a point as a 2-fold circle would. This example, which is easy to generalize to the case of m-fold circles, shows that the stability problem does have its subtleties.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2013.01.014

Very recently, Wang in [15] showed the asymptotic stability of m-fold circles under certain convex small $\frac{2\pi m}{n}$-periodic perturbations as solutions to the curve shortening flow (i.e., for the case when $p = 1$). Other interesting works, besides Wang's, regarding stability of solutions to the curve shortening flow are the by now classical papers of Abresch and Langer [1], and Epstein and Weinstein [6]. We will extend the work of Wang in two ways: we will show asymptotic stability results for the m-fold circle as a solution to the p-curve shortening flow for p any positive integer, and we will provide sharp stabilization estimates for the curvature of solutions to the p-curve shortening flow whose initial data are appropriate small perturbations of an m-fold circle.

To study the stability of m-fold circles as solutions to (1) we will consider the Boundary Value Problem,

$$\begin{cases} \dfrac{\partial k}{\partial t} = k^2\left(k^{p-1}\dfrac{\partial^2 k}{\partial \theta^2} + (p-1)\,k^{p-2}\left(\dfrac{\partial k}{\partial \theta}\right)^2 + \dfrac{1}{p}k^p\right) & \text{in } \left[0, \dfrac{2\pi}{\lambda}\right]\times(0,T) \\ k(\theta,0) = \psi(\theta) \quad \text{on } \left[0,\dfrac{2\pi}{\lambda}\right], \end{cases} \tag{2}$$

$\lambda > \sqrt{\frac{p+2}{p}}$, under periodic boundary conditions, and the initial data ψ is a *strictly positive function*. As it is, (2) has no immediate geometric interpretation. However, when λ is an appropriate rational number, (2) is the evolution equation of the curvature of a curve being deformed via (1) under the assumption that the initial curve satisfies certain symmetries; more precisely, when $\lambda = \frac{n}{m}$, the study of Eq. (2) is equivalent to the study of (1) when the initial data is a perturbation of an m-fold circle under a $\frac{2\pi m}{n}$-periodic perturbation (see also Lemma 2.3 in [12]), and in this case the parameter θ corresponds to the angle formed by the outward unit normal to the curve with respect to the positive x-semiaxis. It is well known that given a strictly positive initial data (2) has a unique solution for a short time (for instance, by identifying the endpoints of $\left[0, \frac{2\pi}{\lambda}\right]$, (2) can be seen as a quasilinear parabolic problem in $\mathbb{S}^1$, so we can use the theory in Section 7 of [13] to prove existence and uniqueness of solutions), and also via the maximum principle it can be shown that this solution blows up in finite time; relevant results on the blow-up behavior of general curve shortening flows can be found in [2,11,12].

The method we will use to prove our stability results was introduced in [5] (inspired by [14]) to study the blow-up behavior of certain nonlinear parabolic equations with periodic boundary conditions, but as the reader will notice, it can be also used to study the stability of certain blow-up profiles, and the regularity of solutions to (2). For instance, as a byproduct of the method we will employ, it can be shown that, under certain conditions on the initial data, solutions to (2) are analytic. So we hope that the reader may find the method used in this paper of independent interest.

The organization of this paper is as follows. In Section 2 we present our main results: Theorem 2.1 (and its restatement for a normalized version of (2): Corollary 2.1), Theorem 2.2, and their application to the stability problem of m-fold circles (Theorem 2.3, whose statement and proof are in Section 2.1). In Section 3, we present the proof of Theorem 2.1. In Section 4, we discuss the exponential stability of the constant steady solution $w \equiv 1$ of the normalized version of (2), and prove Theorem 2.2.

Finally, we want to thank the referee for many useful comments that helped improve the organization and presentation of this paper.

2. Main results

Our results on the behavior of the p-curve shortening flow will follow as a consequence of a few results on the behavior of solutions to (2), that we will promptly describe; but before we state our main results, let us set some definitions and notation. Given $f \in L^2\left(\left[0, \frac{2\pi}{\lambda}\right]\right)$, we write its Fourier expansion as,

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{i\lambda nx} \quad \text{where } \hat{f}(n) = \frac{\lambda}{2\pi}\int_0^{\frac{2\pi}{\lambda}} f(\theta)\, e^{-in\theta}\, d\theta.$$

We will refer to $\hat{f}(n)$ as the n-th Fourier coefficient and to n as the frequency or wave number. We now define the family of seminorms,

$$\|f\|_\beta = \max\left\{\sup_{n\in\mathbb{Z}} |n|^\beta \left|Re\left(\hat{f}(n)\right)\right|, \sup_{n\in\mathbb{Z}} |n|^\beta \left|Im\left(\hat{f}(n)\right)\right|\right\}.$$

As is customary, we define $C^l\left(\left[0, \frac{2\pi}{\lambda}\right]\right)$, $l = 0, 1, 2, \ldots$, as the space of functions with continuous derivatives of order l, equipped with the norm

$$\|f\|_{C^l\left(\left[0,\frac{2\pi}{\lambda}\right]\right)} = \max_{j=0,1,2,\ldots,l}\ \sup_{\theta\in\left[0,\frac{2\pi}{\lambda}\right]} \left|\frac{d^j f(\theta)}{d\theta^j}\right|.$$

Our first result is the following theorem.