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# Well-posedness of nonlocal boundary value problems with integral condition for the system of hyperbolic equations

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ABSTRACT

A nonlocal boundary value problem with integral condition for a system of hyperbolic equations is considered. Relationship with the family of boundary value problems for ordinary differential equations is established. Sufficient and necessary conditions of well-posedness of nonlocal boundary value problems with integral condition for a system of hyperbolic equations are obtained.



## 0. Introduction

In this paper at the domain $\bar{\Omega} = [0, T] \times [0, \omega]$ we consider the following nonlocal boundary value problem with an integral condition for the system of hyperbolic equations with two independent variables

$$\frac{\partial^2 u}{\partial t \partial x} = A(t,x)\frac{\partial u}{\partial x} + B(t,x)\frac{\partial u}{\partial t} + C(t,x)u + f(t,x), \tag{1}$$

$$\begin{aligned} &P_2(x)\frac{\partial u(t,x)}{\partial x}\Big|_{t=0} + P_1(x)\frac{\partial u(t,x)}{\partial t}\Big|_{t=0} + P_0(x)u(t,x)|_{t=0} \\ &\quad + \int_0^T \left[L_2(\tau,x)\frac{\partial u(\tau,x)}{\partial x} + L_1(\tau,x)\frac{\partial u(\tau,x)}{\partial \tau} + L_0(\tau,x)u(\tau,x)\right] d\tau \\ &\quad + S_2(x)\frac{\partial u(t,x)}{\partial x}\Big|_{t=T} + S_1(x)\frac{\partial u(t,x)}{\partial t}\Big|_{t=T} + S_0(x)u(t,x)|_{t=T} = \varphi(x), \quad x \in [0,\omega], \end{aligned} \tag{2}$$

$$u(t,0) = \psi(t), \quad t \in [0,T], \tag{3}$$

where $u(t,x) = col(u_1(t,x), u_2(t,x), \ldots, u_n(t,x))$ is an unknown function, the $(n \times n)$ matrices $A(t,x)$, $B(t,x)$, $C(t,x)$, $L_i(t,x)$, $P_i(x)$, $S_i(x)$, $i = \overline{0,2}$, and $n$-vector functions $f(t,x)$, and $\varphi(x)$ are continuous on $\bar{\Omega}$ and $[0,\omega]$ respectively, and $n$-vector function $\psi(t)$ is continuously differentiable on $[0,T]$.

Let $C(\bar{\Omega}, R^n)$ be the space of continuous on $\bar{\Omega}$ vector functions $u(t,x)$ with the norm $\|u\|_0 = \max_{(t,x)\in\bar{\Omega}} \|u(t,x)\|$, $\|u(t,x)\| = \max_{i=\overline{1,n}} |u_i(t,x)|$.

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The function $u(t,x) \in C(\bar{\Omega}, R^n)$, that has partial derivatives $\frac{\partial u(t,x)}{\partial x} \in C(\bar{\Omega}, R^n)$, $\frac{\partial u(t,x)}{\partial t} \in C(\bar{\Omega}, R^n)$, $\frac{\partial^2 u(t,x)}{\partial t \partial x} \in C(\Omega, R^n)$ is called a *classical solution* to problem (1)–(3) if it satisfies system (1) for all $(t,x) \in \bar{\Omega}$ and the boundary conditions (2) and (3).

Nonlocal boundary value problems for system (1) were researched by numerous authors [6–14,16–31,33]. The most studied problem for hyperbolic equations with mixed partial derivatives is periodic boundary problem. In the works of L. Cesari [10–14] methods for finding periodic solutions of ordinary differential equations were extended to partial differential equations. In J.K. Hale [17], A.K. Aziz [6], A.K. Aziz and A.M. Meyers [9], A.K. Aziz and M.G. Horak [8], A.K. Aziz and S.L. Brodsky [7] the sufficient conditions of the solvability of periodic boundary value problems for linear and nonlinear hyperbolic equations are established by using of the results of L. Cesari. V. Lakshmikantham and S.G. Pandit [27], S.V. Zhestkov [33] studied periodic boundary value problems for linear and nonlinear hyperbolic equations using the method of fixed points for some integral operators. B.I. Ptashnyck [29] researched the conditions of existence of periodic solutions to hyperbolic higher-order equations connected with the problem of small denominators. In the monograph of A.M. Samoilenko, B.P. Tkach [30] the numerical–analytical method is applied to the study of periodic boundary value problems for equations and systems of partial differential equations of hyperbolic type with deviating argument. In the paper of Y.A. Mitropolskii and L.B. Urmancheva [28] the two-point boundary value problem for quasilinear system of hyperbolic equations was studied by the numerical–analytical method. In the works [18–23,25,26] some types of nonlocal boundary problems for the equations and systems of hyperbolic type were examined. In the paper [24] the well-posedness of the initial–boundary problem for linear hyperbolic equations of higher orders with two independent variables, which can be reduced to the system (1) were investigated. Note that of great interest to specialists are nonlocal problems with integral condition for hyperbolic equations [16,31]. In the [1,2] a nonlocal boundary value problem with data on the characteristics of the system of hyperbolic equations without the integral term in the condition (2) is considered. Sufficient conditions are given for the existence and uniqueness of classical solution to problem (1)–(3), when $L_i(t,x) = 0$, $i = \overline{1,3}$ in terms of the initial data. In [3–5] this problem by introducing new functions was reduced to a family of the two-point boundary value problems for ordinary differential equations and functional relations. Coefficient criteria of well-posedness of the considered problem were obtained.

In the present paper the proposed approach is developed to nonlocal boundary value problems with integral condition (1)–(3). Nonlocal problem with the integral condition is reduced to a family of boundary value problems with the integral condition for a system of ordinary differential equations and functional relations. We prove the assertion about equivalence of well-posedness of problem (1)–(3) and well-posedness of the family of boundary value problems with the integral condition for the system of ordinary differential equations. Using the parametrization method [15] the necessary and sufficient conditions of unique solvability of the family boundary value problems with an integral condition for system of ordinary differential equations are established in terms of the initial data. The coefficient criteria of unique solvability of the nonlocal boundary value problem with integral condition (1)–(3) are obtained.

## 1. Problem (1)–(3) and its relationship with the family of boundary value problems for ordinary differential equations

In this section we introduce new unknown functions $v(t,x) = \frac{\partial u(t,x)}{\partial x}$ and $w(t,x) = \frac{\partial u(t,x)}{\partial t}$ and reduce problem (1)–(3) to the equivalent problem

$$\frac{\partial v}{\partial t} = A(t,x)v + F(t,x,w(t,x),u(t,x)), \tag{1.1}$$

$$P_2(x)v(0,x) + S_2(x)v(T,x) + \int_0^T L_2(\tau,x)v(\tau,x)d\tau = \Phi(x,w,u), \quad x \in [0,\omega], \tag{1.2}$$

$$u(t,x) = \psi(t) + \int_0^x v(t,\xi)d\xi, \qquad w(t,x) = \dot{\psi}(t) + \int_0^x \frac{\partial v(t,\xi)}{\partial t}d\xi, \tag{1.3}$$

where $(t,x) \in \bar{\Omega}$, and $F(t,x,w(t,x),u(t,x)) = B(t,x)w(t,x) + C(t,x)u(t,x) + f(t,x)$, $\Phi(x,w,u) = \varphi(x) - P_1(x)w(0,x) - P_0(x)u(0,x) - \int_0^T [L_1(\tau,x)w(\tau,x) + L_0(\tau,x)u(\tau,x)]d\tau - S_1(x)w(T,x) - S_0(x)u(T,x)$.

In the problem (1.1)–(1.3) the condition $u(t,0) = \psi(t)$ is taken into account in relation (1.3).

A triple $\{v(t,x), u(t,x), w(t,x)\}$ of continuous on $\bar{\Omega}$ functions is called a *solution* to problem (1.1)–(1.3) if the function $v(t,x)$ belonging to $C(\bar{\Omega}, R^n)$ has a continuous derivative with respect to $t$ on $\Omega$ and satisfies the one-parameter family of boundary value problems with integral condition for ordinary differential equations (1.1)–(1.2), where the functions $u(t,x)$ and $w(t,x)$ are connected with $v(t,x)$ and $\frac{\partial v(t,x)}{\partial t}$ by the functional relation (1.3).

Let $u^*(t,x)$ be a classical solution of problem (1)–(3). Then the triple $\{v^*(t,x), u^*(t,x), w^*(t,x)\}$, where $v^*(t,x) = \frac{\partial u^*(t,x)}{\partial x}$, $w^*(t,x) = \frac{\partial u^*(t,x)}{\partial t}$, is a solution to problem (1.1)–(1.3). Conversely, if a triple $\{\widetilde{v}(t,x), \widetilde{u}(t,x), \widetilde{w}(t,x)\}$ is a solution to problem (1.1)–(1.3), then $\widetilde{u}(t,x)$ is a classical solution to problem (1)–(3).

For fixed $w(t,x)$, $u(t,x)$ in problem (1.1)–(1.3) it is necessary to find a solution to a one-parameter family of boundary value problems with an integral condition for system of ordinary differential equations.