J. Math. Anal. Appl. 402 (2013) 201–206

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Hamilton's gradient estimates and Liouville theorems for porous medium equations on noncompact Riemannian manifolds

Xiaobao Zhu

Department of Mathematics, School of Information, Renmin University of China, Beijing 100872, People's Republic of China

ARTICLE INFO

Article history:
Received 9 April 2012
Available online 23 January 2013
Submitted by Andrea Cianchi

Keywords:
Gradient estimate
Porous medium equation
Liouville theorem

ABSTRACT

Let M be a complete noncompact Riemannian manifold of dimension n. In this paper, we derive a local gradient estimate for positive solutions of porous medium equation

$$\partial_t u = \Delta u^\alpha, \quad 1 < \alpha < 1 + \frac{1}{\sqrt{2n} + 1}$$

on $M \times (-\infty, 0]$. We also obtain a theorem of Liouville type for positive solutions of the porous medium equation.

1. Introduction

In this paper we study the porous medium equation (PME for short)

$$\partial_t u = \Delta u^\alpha, \tag{1.1}$$

where $\alpha \in \left(1, 1 + \frac{1}{\sqrt{2n}+1}\right)$. PME appears in the description of differential natural phenomenon, like processes involving fluid flow, heat transfer or diffusion, the flow of an isentropic gas through a porous medium, groundwater infiltration, heat radiation in plasmas and so on. We refer the reader to the new book by Vázquez [8] and the references therein for more about PME.

As a nonlinear problem, the mathematical theory of PME is based on a priori estimates. In 1979, Aronson and Bénilan obtained a celebrated second-order differential inequality of the form [1]

$$\sum_i \frac{\partial}{\partial x^i}\left(m u^{m-2} \frac{\partial u}{\partial x^i}\right) \geq -\frac{\kappa}{t}, \quad \kappa := \frac{n}{n(m-1)+2}$$

which applies to all positive solutions of (1.1) defined on the whole Euclidean space on the condition that $m > 1 - \frac{2}{n}$.

There are few results about PME on manifolds. In 2008, Lu, Ni, Vázquez and Villani studied the PME on manifolds [6], they got a local Aronson–Bénilan estimate. We do not state their result here. What we will do in this paper is to get an Hamilton's gradient estimates. We will use $\text{Ricci}(\mathbf{M})$ to denote the Ricci curvature of $\mathbf{M}$ throughout this paper. First, let we recall what Hamilton's result is:

Theorem (*Hamilton [3]*)**.** *Let $\mathbf{M}$ be a closed Riemannian manifold of dimension $n \geq 2$ with $\text{Ricci}(\mathbf{M}) \geq -k$ for some $k \geq 0$. Suppose that u is any positive solution to the heat equation with $u \leq M$ for all $(x, t) \in \mathbf{M} \times (0, \infty)$. Then*

$$\frac{|\nabla u|^2}{u^2} \leq \left(\frac{1}{t} + 2k\right) \log \frac{M}{u}.$$

E-mail addresses: math0523@163.com, zhuxiaobao@ruc.edu.cn.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2013.01.018

Hamilton's estimate tell us, when the temperature is bounded, we can compare the temperature of two different points at the same time.

In 2006, P. Souplet and Qi S. Zhang [7] generalized Hamilton's estimate to complete noncompact Riemannian manifolds. In 2007, B. Kotschwar [4] used Hamilton's estimate and got a global gradient estimate for heat kernels on complete noncompact manifolds. In 2010, M. Bailesteanu, X. Cao and A. Pulemotov [2] generalized Souplet and Zhang's result to Ricci flow. Also in 2010, on complete noncompact Riemannian manifolds, J. Wu [9] obtained a localized Hamilton-type gradient estimate for the positive smooth bounded solutions to the nonlinear diffusion equation $u_t = \Delta u - \nabla\phi \cdot \nabla u - au\log u - bu$, where ϕ is a C^2 function, and $a \neq 0$ and b are two real constants. In 2011, the author studied the fast diffusion equation (with $\alpha \in \left(1-\frac{2}{n}, 1\right)$ in Eq. (1.1)) on a noncompact complete Riemannian manifold, and derived Hamilton's gradient estimates for positive solutions [10].

In this paper, we consider positive solutions for PME (1.1). Applying the same techniques that were used for the heat equation, we derive a Hamilton type estimate for the PME, similar in nature to the ones obtained in the papers discussed above. Let $1 < \alpha < 1 + \frac{1}{\sqrt{2n+1}}$ through this paper. Note that the *pressure* $v := \frac{\alpha}{\alpha-1}u^{\alpha-1}$ satisfies

$$\partial_t v = (\alpha-1)v\Delta v + |\nabla v|^2. \tag{1.2}$$

Our main result is the following:

Theorem 1.1 (*Gradient Estimates*). *Let* $\mathbf{M}$ *be a Riemannian manifold of dimension* $n \geq 2$ *with* $\mathrm{Ricci}(\mathbf{M}) \geq -k$ *for some* $k \geq 0$. *Suppose that* v *is any positive solution to the Eq. (1.2) in* $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset \mathbf{M} \times (-\infty, \infty)$. *Suppose also that* $v \leq M$ *in* $Q_{R,T}$. *Then there exists a constant* $C = C(\alpha, \mathbf{M})$ *such that*

$$v^{\frac{1}{4}\frac{2-\alpha}{\alpha-1}}|\nabla v| \leq CM^{1+\frac{1}{4}\frac{2-\alpha}{\alpha-1}}\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k}\right)$$

in $Q_{\frac{R}{2},\frac{T}{2}}$.

As an application, we get the following Liouville type theorem:

Corollary 1.2 (*Liouville Type Theorem*). *Let* $\mathbf{M}$ *be a complete, noncompact manifold of dimension* n *with nonnegative Ricci curvature. Let* u *be a positive ancient solution to the Eq. (1.1) with* $1 < \alpha < 1 + \frac{1}{\sqrt{2n+1}}$ *such that* $u(x,t) = o\left(\left[d(x) + \sqrt{|t|}\right]^{\frac{4}{3\alpha+2}}\right)$ *near infinity. Then* u *is a constant.*

2. Proof of Theorem 1.1

Let $w \equiv \frac{|\nabla v|^2}{v^\beta}$, β is a real number to be determined.

We want to estimate the quantity w in this paper, and we will begin with deriving an equation for it. First, using (1.2) we have

$$\begin{aligned} w_t &= \frac{2v_iv_{it}}{v^\beta} - \beta\frac{v_i^2v_t}{v^{\beta+1}} \\ &= \frac{2v_i\left((\alpha-1)v\Delta v + |\nabla v|^2\right)_i}{v^\beta} - \beta\frac{v_i^2\left((\alpha-1)v\Delta v + |\nabla v|^2\right)}{v^{\beta+1}} \\ &= 2(\alpha-1)\frac{v_i^2v_{jj}}{v^\beta} + 2(\alpha-1)\frac{v_iv_{jji}}{v^{\beta-1}} + 4\frac{v_iv_{ij}v_j}{v^\beta} \\ &\quad - \beta(\alpha-1)\frac{v_i^2v_{jj}}{v^\beta} - \beta\frac{v_i^2v_j^2}{v^{\beta+1}}, \end{aligned} \tag{2.1}$$

$w_j = \frac{2v_iv_{ij}}{v^\beta} - \beta\frac{v_i^2v_j}{v^{\beta+1}}$.

$$w_{jj} = \frac{2v_{ij}^2}{v^\beta} + \frac{2v_iv_{ijj}}{v^\beta} - 4\beta\frac{v_iv_{ij}v_j}{v^{\beta+1}} - \beta\frac{v_i^2v_{jj}}{v^{\beta+1}} + \beta(\beta+1)\frac{v_i^2v_j^2}{v^{\beta+2}}. \tag{2.2}$$

By (2.1) and (2.2),

$$\begin{aligned} (\alpha-1)v\Delta w - w_t &= 2(\alpha-1)\frac{v_{ij}^2}{v^{\beta-1}} + 2(\alpha-1)\frac{v_iv_{ijj}}{v^{\beta-1}} - 4\beta(\alpha-1)\frac{v_iv_{ij}v_j}{v^\beta} \\ &\quad - \beta(\alpha-1)\frac{v_i^2v_{jj}}{v^\beta} + \beta(\beta+1)(\alpha-1)\frac{v_i^2v_j^2}{v^{\beta+1}} \end{aligned}$$