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# Variation of normal derivatives of Green functions

Charles Z. Martin

*Department of Mathematics, University of California, Santa Barbara, CA 93106, United States*

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ABSTRACT

We derive a variational formula for the outward normal derivative of the Green function for the Schrödinger and Laplace–Beltrami operators, viewed as perturbations of the Laplacian. As an application we begin to characterize elliptic growth—the growth of a domain pushed outward by its own Green function.



## 1. Introduction

The outward normal derivative of the Green function of a domain is of central importance to boundary-value problems associated to the underlying elliptic operator. Given a domain $D \subset \mathbb{C}$, a point $w \in D$, and an elliptic differential operator $L$, we define the Green function $g_w : \overline{D} - \{w\} \to \mathbb{R}$ as the solution (if it exists) to

$$\begin{cases} Lg_w = \delta_w & \text{in } D \\ g_w = 0 & \text{on } \partial D, \end{cases}$$

where $\delta_w$ denotes the Dirac delta supported at $w$. Occasionally we will denote $g_w(z)$ by $g(z, w)$, emphasizing the functional dependence upon both inputs. The existence of a Green function for a domain is tied to the regularity of the boundary. We will work only with smooth and analytic boundaries, which guarantees existence of Green functions and allows use of Green's theorem.

The Green function is used to solve the inhomogeneous boundary-value problem for $L$ with zero boundary data. That is, a solution to

$$\begin{cases} Lu = f & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases}$$

is given by

$$u(w) = \int_D g_w f \, dA.$$

Of equal importance is the homogeneous boundary-value problem

$$\begin{cases} Lu = 0 & \text{in } D \\ u = f & \text{on } \partial D. \end{cases} \qquad (1)$$

*E-mail address:* cmart07@math.ucsb.edu.

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In some situations, the Green function can be used to extract a solution to this problem as well. In the case of the Laplacian, $L = \Delta$, a solution to (1) is given by

$$u(w) = \int_{\partial D} f \partial_n g_w \, ds,$$

where $\partial_n$ denotes the outward normal derivative. The integral kernel $P_\zeta(w) = \partial_n g_w(\zeta)$ is called the Poisson kernel of the region $D$, and for simplicity of notation we sometimes use $P_\zeta(w)$ in place of $\partial_n g_w(\zeta)$, especially when we wish to view $\zeta$ as a fixed parameter.

If instead we consider an operator in divergence form, $L = \nabla\lambda\nabla$, then a solution to (1) is given by

$$u(w) = \int_{\partial D} f \lambda \partial_n g_w \, ds.$$

The normal derivative of the Green function also appears in the dynamics of moving boundaries. In [1] the authors define a process wherein a domain grows with a boundary velocity determined by the outward derivative of its own Green function. This so-called elliptic growth is used to model real processes such as electrodeposition, crystal formation, and quasi-static fluid flowing through a porous medium. When the underlying operator is the Laplacian the growth is commonly called Laplacian growth or Hele–Shaw flow—see [2] for a survey. When the underlying operator is more general however, much less work has been done. It is known for example that lemniscates cannot survive a Laplacian growth process (see [3]), but a carefully-chosen elliptic growth might preserve them. To approach such problems we need an understanding of how the Green function depends upon the operator.

Compared to that of a general elliptic operator, the Green function of the Laplacian is well-understood. In [4] the author develops formulas for Green functions under various perturbations of the Laplacian; here we follow the same paradigm, studying the normal derivative of the Green function under a perturbation of the operator.

## 2. Variation of the normal derivative

### 2.1. Schrödinger

Our starting point is the variational formula derived in [4]. Consider a bounded domain $D$ with smooth, analytic boundary, a positive function $u \in C^\infty(\overline{D})$ and $z, w \in D$. For $\epsilon > 0$ the Green function $g_w^*$ of $\Delta - \epsilon u$ satisfies

$$g_w^*(z) = g_w(z) + \epsilon \int_D u(\xi) g(z,\xi) g(w,\xi) \, dA(\xi) + o(\epsilon) \tag{2}$$

as $\epsilon \to 0$, where $g_w$ denotes the Green function for the Laplacian and the error term converges uniformly in $z$ for each fixed $w$. Furthermore, a full series expansion is given by

$$g_w^* = \sum_{n=0}^{\infty} \epsilon^n (TM)^n g_w,$$

where we have used the operators $M : \varphi \mapsto u\varphi$ and

$$T : \varphi \mapsto \int_D g_w \varphi \, dA.$$

To derive a similar formula for $\partial_n g_w^*$ our major tool is the following lemma.

**Lemma 1.** *Let $D \subset \mathbb{C}$ be a bounded domain with smooth, analytic boundary and $f \in C^1(\overline{D}) \cap C^2(D)$. Suppose further that $f = 0$ on $\partial D$. Then for $\zeta \in \partial D$ we have*

$$\frac{\partial f}{\partial n}(\zeta) = \int_D \Delta f(z) P(z,\zeta) \, dA(z),$$

*where $P(z,\zeta) = \partial_n g_z(\zeta)$ is the Poisson kernel of $D$.*

**Proof.** Let $A \subset \partial D$ be measurable with respect to arc length on $\partial D$. We define an auxiliary function $u$ which solves the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = \chi_A & \text{on } \partial D \end{cases}$$

where $\chi_A$ denotes the characteristic function of $A$. From the Green identity

$$\int_{\partial D} \left( u \frac{\partial f}{\partial n} - f \frac{\partial u}{\partial n} \right) ds = \int_D (u\Delta f - f\Delta u) \, dA \tag{3}$$