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New Jensen-type inequalities

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ABSTRACT

We develop a new framework for the Jensen-type inequalities that allows us to deal with functions that are not necessarily convex and Borel measures that are not necessarily positive.

The important role played by the classical inequality of Jensen in probability theory, economics, statistical physics, information theory etc is well known. See [5,6]. In recent years, a number of authors have noticed the possibility of extending this inequality to the framework of functions that are mixed convex (in the sense of the existence of one inflection point). See [1,2,4]. In all of these papers one assumes that both the function and the measure under consideration verify certain conditions of symmetry. However, the inequality of Jensen is much more general, as the following simple remark shows. Suppose that K is a convex subset of the Euclidean space $\mathbb{R}^N$ carrying a Borel probability measure μ. Then every μ-integrable function $f : K \to \mathbb{R}$ that admits a supporting hyperplane at the barycenter of μ,

$$b_\mu = \int_K x d\mu(x), \tag{B}$$

verifies the Jensen inequality

$$f(b_\mu) \leq \int_K f(x) d\mu(x). \tag{J}$$

Indeed, the existence of a supporting hyperplane at b_μ is equivalent to the existence of an affine function $h(x) = \langle x, v\rangle + c$ such that

$$f(b_\mu) = h(b_\mu) \quad \text{and} \quad f(x) \geq h(x) \quad \text{for all } x \in K.$$

Then

$$f(b_\mu) = h(b_\mu) = h\left(\int_K x d\mu(x)\right) = \int_K h(x) d\mu(x) \leq \int_K f(x) d\mu(x).$$

As is well known, the convexity assures the existence of a supporting hyperplane at each interior point. See [5, Theorem 3.7.1, p. 128]. This explains why Jensen's inequality works nicely in that context. The aim of our paper is to extend the validity of Jensen's inequality outside mixed convexity and also outside the framework of Borel probability measures.

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In order to make our approach easily understandable we will restrict ourselves to the case of functions of one real variable. However, most of our results can be extended easily to higher dimensions, by replacing the usual intervals by N-dimensional intervals and symmetry with respect to a point by symmetry with respect to a hyperplane. See Example 3.

We start with the following version of the Jensen inequality for mixed convex functions that discards any assumption on the symmetry of the measure involved.

Theorem 1. *Suppose that f is a real-valued function defined on an interval $[a, b]$ and c is a point in $[a, \frac{a+b}{2}]$ such that:*

(i) $f(c-x)+f(c+x)=2f(c)$ *whenever* $c \pm x \in [a, b]$;
(ii) $f|_{[c,b]}$ *is convex.*

Then

$$f\left(b_{\mu}\right) \leq \int_a^b f(x)d\mu(x),$$

for every Borel probability measure μ on $[a, b]$ whose barycenter lies in the interval $[2c-a, b]$.

The last inequality works in the reverse direction when $f|_{[c,b]}$ is concave.

Proof. The case where $c = a$ is covered by the classical inequality of Jensen, so we may assume that $c \in (a, \frac{a+b}{2})$. In this case the point $2c-a$ is interior to $[a, b]$. By our hypotheses, the barycenter b_μ lies in the interval $[2c-a, b]$. If $b_\mu = b$, then $\mu = \delta_b$ and the conclusion of Theorem 1 is clear. If b_μ is interior to $[a, b]$, we will denote by h the affine function joining the points $(a, f(a))$ and $(2c-a, f(2c-a))$ and we will consider the function

$$g(x) = \begin{cases} h(x) & \text{if } x \in [a, 2c-a] \\ f(x) & \text{if } x \in [2c-a, b]. \end{cases} \tag{1}$$

Clearly, g is convex and this fact motivates the existence of a support line ℓ of g at b_μ. See [5, Lemma 1.5.1, p. 30]. Since $h \geq f$, then necessarily ℓ is a support line at b_μ also for f. By a remark above, this ends the proof. □

A useful consequence of Theorem 1 in the case of absolutely continuous measures is as follows:

Corollary 1. *Suppose that $f : [-b, b] \to \mathbb{R}$ is an odd function whose restriction to $[0, b]$ is convex and $p : [-b, b] \to [0, \infty)$ is a nondecreasing function that does not vanish on $(-b/3, b]$. Then for every $a \in [-b/3, b)$,*

$$f\left(\frac{1}{\int_a^b p(x)dx}\int_a^b xp(x)dx\right) \leq \frac{1}{\int_a^b p(x)dx}\int_a^b f(x)p(x)dx.$$

Proof. The case where $a \geq 0$ is covered by the classical inequality of Jensen.

If $a < 0$, then

$$\begin{aligned}\int_a^b (x+a)p(x)dx &\geq \int_a^{-3a}(x+a)p(x)dx \\ &= \int_a^{-a}(x+a)p(x)dx + \int_{-a}^{-3a}(x+a)p(x)dx \\ &\geq \int_a^{-a}(x+a)p(x)dx + p(-a)\int_{-a}^{-3a}(x+a)dx \\ &= \int_a^{-a}(x+a)p(x)dx - p(-a)\int_a^{-a}(x+a)dx \\ &= \int_a^{-a}(x+a)\left(p(x)-p(-a)\right)dx \geq 0,\end{aligned}$$

and thus Theorem 1 applies. □

An inspection of the argument of Corollary 1 shows that the monotonicity hypothesis on p can be relaxed by asking only for the integrability of p and the fact that $p(x) \leq p(-a) \leq p(y)$ for all x and y with $x \leq -a \leq y$. However, simple examples show that the restriction $a \in [-b/3, b]$ in Corollary 1 cannot be dropped.

Consider now the discrete version of Theorem 1.

Corollary 2. *Suppose that f is a real-valued function defined on an interval I that contains the origin such that $f|_{I\cap[0,\infty)}$ is a convex function and $f(-x) = -f(x)$ whenever x and $-x$ belong to I. Then for every family of points $a_1, \ldots, a_n$ of I and every*