



Hilbert spaces of entire Dirichlet series and composition operators

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ABSTRACT

The aim of this paper is to introduce Hilbert spaces of entire Dirichlet series with real frequencies and consider composition operators on these spaces. We establish necessary and sufficient conditions for such series to have Ritt order zero, as well as to have finite logarithmic orders. This allows us to apply the Polya theorem on composition of entire functions to consideration of composition operators on the Hilbert spaces of entire Dirichlet series. In particular, criteria for action, boundedness, compactness and compact difference of such operators are obtained.

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1. Introduction

Let us consider Dirichlet series with real frequencies

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}, \quad a_n, z \in \mathbb{C}, \quad (1.1)$$

where $0 \leq (\lambda_n) \uparrow \infty$ is a given sequence of real numbers.

As is well known, if we let

$$L = \limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n},$$

and set

$$D = \limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n},$$

then in case $L < \infty$, the series (1.1) represents an entire function in $\mathbb{C}$ if and only if $D = -\infty$ (see, e.g., [4,9]).

Throughout this paper, the condition $L < \infty$ is always supposed to hold.

Let $\mathcal{H}$ be a space of holomorphic functions on a set $G \subseteq \mathbb{C}$. If an analytic function φ maps G into itself, the *composition operator* C_φ is a linear operator defined by the rule

$$(C_\varphi f)(z) = f \circ \varphi(z), \quad z \in G, f \in \mathcal{H}.$$

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Composition operators have been investigated on various spaces of holomorphic functions of one variable, as well as in higher dimensions. We refer the reader to the excellent monographs [2,14] for detailed information. In particular, an extensive study of composition operators was carried out on spaces of classical Dirichlet series, that is for the case when the frequencies $\lambda_n = \log n$, $\forall n \geq 1$ (see, e.g., [1,3,5]).

The aim of the present paper is to study composition operators on Hilbert spaces of entire Dirichlet series. It should be noted that we make use of Pólya's result [11], and therefore, the three notions of order for Dirichlet series (ordinary, Ritt and logarithmic orders) play an important role in our approach. It seems this topic has never been treated before.

The structure of the papers is as follows. In Section 2 we introduce some classes of Hilbert spaces of entire Dirichlet series which arise in depending on the weight sequences. Several auxiliary properties needed in the sequel are provided. Section 3 deals with a study of when the series of those Hilbert spaces have Ritt order zero, and finite logarithmic orders. The results obtained in this section allow to use the Polya result on composition of entire functions of finite ordinary order in the main part of the paper. In Section 4 we study various problems of composition operators on Hilbert spaces of entire Dirichlet series whose elements have Ritt order zero and finite logarithmic orders. The criteria for action, boundedness, compactness and compact differences are obtained.

We note that some of our results were announced in [6].

2. Hilbert space of entire Dirichlet series

Let $\Lambda := (\lambda_n)$, with $0 \leq (\lambda_n) \uparrow \infty$ satisfying condition $L < \infty$, be given. Consider the normed space of entire Dirichlet series

$$\mathcal{H}(E) := \left\{ f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} : (a_n) \in E \right\},$$

where

$$E = \left\{ (a_n) : \limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n} = -\infty, \text{ or } \lim_{n \rightarrow \infty} |a_n|^{1/\lambda_n} = 0 \right\}.$$

The norm in this space is defined by the inner product

$$\langle f, g \rangle := \sum_{n=1}^{\infty} a_n \bar{b}_n, \quad f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}, \quad g(z) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n z}.$$

In [7], different properties of composition operators on a subspace $\mathcal{H}(E, \rho, 0)$ of the space $\mathcal{H}(E)$, such as action, boundedness, were considered.

It is important to note that the space $\mathcal{H}(E, \rho, 0)$ as well as $\mathcal{H}(E)$ are never complete with respect to the norm above, and hence other important properties of composition operators (compactness, compact differences, ...) cannot be considered.

Then a natural question to ask is: how do we define subspaces of $\mathcal{H}(E)$ that can be Hilbert spaces? We are going to consider this question.

To each sequence of real positive numbers $\beta = (\beta_n)$ we associate the following sequence space

$$\ell_{\beta}^2 = \left\{ a = (a_n) : \|a\| = \left(\sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 \right)^{1/2} < \infty \right\},$$

which is a Hilbert (sequence) space with the inner product

$$\langle a, b \rangle = \sum_{n=1}^{\infty} a_n \bar{b}_n \beta_n^2, \quad \forall (a_n), (b_n) \in \ell_{\beta}^2, \quad (2.1)$$

(see, e.g., [15]).

First we note the following elementary result (partially stated earlier in [8] under the condition $L = 0$ with complex frequencies), which is used very often in our discussion.

Lemma 2.1. *Let $0 \leq (\lambda_n) \uparrow \infty$ be given. The following conditions are equivalent:*

- (a) $\limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} = L < \infty$.
- (b) $\forall r > L : \sum_{n=1}^{\infty} e^{-r \lambda_n} < \infty$.
- (c) $\exists \rho > 0 : \sum_{n=1}^{\infty} e^{-\rho \lambda_n} < \infty$.
- (d) $\limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} \leq \rho$.

The above conditions imply

- (e) $\forall r < L : \sum_{n=1}^{\infty} e^{-r \lambda_n} = \infty$.

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