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Discontinuous piecewise quadratic Lyapunov functions for planar piecewise affine systems

Najmeh Eghbal *, Naser Pariz, Ali Karimpour

Electrical Engineering Department, Ferdowsi University of Mashhad, P.O. Box 91775-1111, Iran

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ABSTRACT

For planar piecewise affine systems, this paper proposes sufficient stability conditions based on discontinuous Lyapunov functions. The monotonicity condition for discontinuous functions at switching instants is presented based on the behavior of state trajectories on the switching surfaces. First, the stability conditions are derived for a typical multiple Lyapunov function and then these conditions are formulated as a set of linear matrix inequalities for piecewise quadratic Lyapunov functions. The implementation of the proposed method is illustrated by an example.

1. Introduction

Hybrid systems characterize classes of dynamical systems which consist of both continuous and discrete dynamics. Since hybrid models can adequately describe the behavior of many physical systems, there is a great interest in studying hybrid systems. The class of piecewise affine (PWA) systems is a general and well-studied class of hybrid systems. It consists of a set of affine subsystems and a switching law that selects the active subsystem based on the sub-region that the states of the system belong to. A large class of nonlinear systems in engineering applications can be approximated by PWA systems [1]. Also, PWA systems are equivalent to several classes of hybrid systems [2,3], whereas PWA systems allow using tractable mathematical tools for analysis and synthesis. Thus, piecewise affine systems provide a powerful means for analysis and synthesis of many nonlinear systems. A wide range of PWA systems is continuous. To name a few, the approximated PWA systems obtained from modeling a nonlinear system are continuous [4–6]. Also some PWA systems which describe physical nonlinearities like dead-zone, saturation and hysteresis are continuous. In the recent decade, the stability issues of PWA systems have drawn a lot of attention [7–10]. Due to hybrid behavior of PWA systems, the analysis of even simple PWA systems can lead to an NP hard problem [11]. The existence of a single quadratic Lyapunov function for all subsystems of PWA system can ensure the quadratic stability of the switched system. In order to find less conservative stability conditions for hybrid systems, the theorem of multiple Lyapunov functions is presented in [12]. As the behavior of the system at switching instants must be known priori, the application of this theorem is difficult. However, for PWA systems, this theorem is relaxed by posing the continuity condition of the Lyapunov function on the boundaries of sub-regions. In the last decades, several multiple Lyapunov functions have been proposed based on the mentioned relaxation method. To name a few, in [13], piecewise quadratic (PWQ) Lyapunov functions are introduced for continuous-time PWA systems. [14–16] present piecewise affine Lyapunov functions and in [17], sufficient conditions for the stability of piecewise linear (PWL) systems are proposed using homogeneous polynomial Lyapunov functions. Also an extension for discontinuous PWQ Lyapunov functions is presented in [18], however it is only for PWA systems in which the switching surfaces are traversed by trajectories of the system in the known directions.

* Corresponding author.
E-mail address: najmeh.eghbal@gmail.com (N. Eghbal).

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This paper presents stability conditions based on discontinuous multiple Lyapunov functions, for planar continuous PWA systems. As the set of continuous functions is the subset of the set of discontinuous functions, it is clear that by relaxing the continuity of Lyapunov functions on switching surfaces, the search for Lyapunov functions is done in a bigger set of functions and so the conservativeness in stability analysis decreases considerably. In the proposed method, for monotonicity of the Lyapunov function at switching instants, one does not need to have any information on the trajectories of the system at switching surfaces and the monotonicity condition is only presented based on the vector field of the system. For discontinuous PWQ Lyapunov functions, the sufficient conditions for the stability of PWA systems are formulated as linear matrix inequalities (LMIs) which can be solved using a standard LMI solver.

The remainder of this paper is organized as follows. The notation used throughout the text and some preliminary results are presented in Section 2. Section 3 deals with the discontinuous functions and monotonicity of these functions at switching surfaces. The conditions for the stability of hybrid systems via discontinuous Lyapunov functions are given in Section 4. These conditions are formulated for discontinuous PWQ Lyapunov functions in Section 5. Section 6 is dedicated to Simulation results and finally, some concluding remarks are drawn in Section 7.

2. Notation and preliminaries

Definition 1 (*Polyhedron*)**.** A convex set X in the d-dimensional space which defined as $X = \left\{x \in R^d | a^T x \geq b\right\}$ is called a polyhedron with $a \in R^{d\times n}$ and $b \in R^n$. The mentioned inequality means the element-wise inequality.

Definition 2 (*Polyhedral Partition*)**.** A collection of polyhedron $X_i \subseteq X, i \in I \subset N$, is the polyhedral partition of the polyhedron X, if and only if $\bigcup_{i\in I} \bar{X}_i = X$ and $X_i \bigcap X_j = \phi,\ \forall i, j \in I,\ i \neq j.$

The state space equations describing a planar PWA system in $X \subset R^2$ are

$$\dot{x}(t) = A_i x(t) + a_i \quad x \in \bar{X}_i,\ i \in I \tag{1}$$

where $x(t) \in X$ is the state vector and A_i and a_i are constant matrix/vector of suitable dimensions. Let I be the set of mode indices and $\{X_i\}_{i\in I}$ be a polyhedral partition of X into a number of cells with $\bigcup_{i\in I} \bar{X}_i = X$ and $X_i \bigcap X_j = \phi,\ \forall i,\ j \in I,\ i \neq j$ where $\bar{X}_i$ denotes the closure of X_i. Suppose the countable set I, *card* $\{I\}$ denotes the cardinality of I. It is assumed that for $i \in I_0, I_0 = \left\{i \in I : 0 \in \bar{X}_i\right\}$, the origin is the only equilibrium point for the corresponding subsystem and for $i \notin I_0$, the corresponding subsystem has not any equilibrium point in X_i.

Since the cells are polyhedron, we have,

$$\bar{X}_i = \left\{x \in R^2 : E_i x \geq e_i\right\}, \quad i \in I \tag{2}$$

where E_i and e_i are constant matrix/vector. A parametric description of the boundary between two regions X_i and X_j where $\bar{X}_i \bigcap \bar{X}_j \neq \phi$, can be described as

$$\bar{X}_i \bigcap \bar{X}_j \subseteq \left\{x |\ x = F_{ij} s + f_{ij},\ s \in R\right\}. \tag{3}$$

By setting $\bar{x} = \begin{bmatrix} x^T & 1 \end{bmatrix}^T$, (1)–(3) can be written in the more compact form

$$\dot{\bar{x}} = \bar{A}_i \bar{x} \quad x \in \bar{X}_i \tag{4}$$

$$\bar{X}_i = \left\{x \in R^2 : \bar{E}_i \bar{x} \geq 0\right\} \tag{5}$$

$$\bar{X}_i \bigcap \bar{X}_j \subseteq \left\{x |\ \bar{x} = \bar{F}_{ij} \bar{s},\ \bar{s} = [s^T \quad 1]^T,\ s \in R\right\} \tag{6}$$

where

$$\bar{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix}, \qquad \bar{E}_i = \begin{bmatrix} E_i & -e_i \end{bmatrix}$$

$$\bar{F}_{ij} = \begin{bmatrix} F_{ij} & f_{ij} \\ 0 & 1 \end{bmatrix}.$$

For boundaries defined in (3), if $F_{ij} \neq 0$, then the boundary is a part of a line and if $F_{ij} = 0$, then the boundary is a point. For $\bar{X}_i \bigcap \bar{X}_j \neq \phi$ with $F_{ij} \neq 0$ (linear boundary), we can define $\bar{C}_{ij} = \begin{bmatrix} C_{ij} & c_{ij} \end{bmatrix}$and $S_{ij} = \left\{x |\ \bar{C}_{ij} \bar{x} = 0\right\}$ in which C_{ij} is the normal vector of S_{ij} (a vector perpendicular to S_{ij}) with the direction from X_i to X_j such that $\bar{X}_i \bigcap \bar{X}_j \subseteq S_{ij}$.

Consider the hybrid system:

$$\dot{x}(t) = f_i(x) \quad x \in \bar{X}_i,\ i \in I \tag{7}$$

where f_i is a continuous function and X_i is a polyhedral region which belongs to R^2. In the following, at first, discontinuous Lyapunov functions and the monotonicity condition of discontinuous functions on boundaries are discussed for system (7) and then these results are regenerated for PWA system (4). Note that the hybrid system (7) is assumed to be continuous, so based on the continuity of the system, the vector fields of the neighbor subsystems on the common boundary are the same or $f_i(x) = f_j(x)$ for $x \in \bar{X}_i \bigcap \bar{X}_j$.