J. Math. Anal. Appl. 396 (2012) 84–92

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Non-central Dirichlet Type 2 distributions on symmetric matrices

Mohamed Ben Farah

Laboratory of Probability and Statistics, Sfax University, B.P.802, Sfax, Tunisia

ARTICLE INFO

Article history:
Received 21 October 2011
Available online 9 June 2012
Submitted by Vladimir Pozdnyakov

Keywords:
Asymptotic
Bessel function
Confluent hypergeometric function
Kampé de Fériet's function
Non-central Dirichlet distribution
Non-central Wishart distribution

ABSTRACT

In this paper, we introduce the non-central Dirichlet Type 2 distribution on symmetric matrices as an extension of the real non-central Dirichlet Type 2 distribution defined by non-central gamma distribution. We also establish some properties concerning this distribution such as marginal and conditional distributions, distribution of partial sums, moments and asymptotic results.

1. Introduction

A random variable Y is said to have a non-central gamma distribution with parameters p (>0), θ (>0) and δ (≥ 0), denoted by $Ga(p, \theta, \delta)$, if its probability density function (pdf) is given by

$$\{\theta^p \Gamma(p)\}^{-1} \exp\left(-\delta - \frac{y}{\theta}\right) y^{p-1} {}_0F_1\left(p, \frac{\delta y}{\theta}\right) \mathbf{1}_{(0,\infty)}(y)$$

where $\Gamma(.)$ is the gamma function defined by

$$\Gamma(p) = \int_0^\infty \exp(-x) x^{p-1} dx,$$

${}_0F_1$ is the Bessel function defined by

$${}_0F_1(a; x) = \sum_{k=0}^{\infty} \frac{x^k}{(a)_k k!},$$

and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1),$$

for $n = 1, 2, \ldots$, and $(a)_0 = 1$.

Note that if $\delta = 0$, the non-central gamma distribution reduces to the gamma distribution.

Let $Y_1, \ldots, Y_{n+1}$ be independent random variables, $Y_i \sim Ga(p_i, \theta, \delta_i)$ and define

$$X = (X_1, \ldots, X_n) = \left(\frac{Y_1}{Y_{n+1}}, \ldots, \frac{Y_n}{Y_{n+1}}\right).$$

E-mail address: mohamed.benfarah@isbs.rnu.tn.

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doi:10.1016/j.jmaa.2012.05.069

Then the distribution of X is called the non-central Dirichlet Type 2 distribution with parameters $(p_1, \ldots, p_n; p_{n+1}; \delta_1, \ldots, \delta_n; \delta_{n+1})$ and is denoted by $D^2(p_1, \ldots, p_n; p_{n+1}; \delta_1, \ldots, \delta_n; \delta_{n+1})$. Its pdf is given by

$$\frac{\Gamma_r\left(\sum_{i=1}^{n+1} p_i\right)}{\prod_{i=1}^{n+1} \Gamma_r(p_i)} \exp\left(-\sum_{i=1}^{n+1}\delta_i\right) \prod_{i=1}^{n} x_i^{p_i-1}\left(1+\sum_{i=1}^{n} x_i\right)^{-\sum_{i=1}^{n+1} p_i}$$
$$\times \Psi_2^{(n+1)}\left(\sum_{i=1}^{n+1} p_i; p_1, \ldots, p_{n+1}; \frac{\delta_1 x_1}{1+\sum_{i=1}^{n} x_i}, \ldots, \frac{\delta_n x_n}{1+\sum_{i=1}^{n} x_i}, \frac{\delta_{n+1}}{1+\sum_{i=1}^{n} x_i}\right), \tag{1.1}$$

where $x_i > 0, i = 1, \ldots, n$ and $\Psi_2^{(m)}$ is the confluent hypergeometric function in m variables $z_1, \ldots, z_m$ defined by

$$\Psi_2^{(m)}(a; c_1, \ldots, c_m; z_1, \ldots, z_m) = \sum_{j_1,\ldots,j_m=0}^{\infty} \frac{(a)_{j_1+\cdots+j_m} z_1^{j_1} \ldots z_m^{j_m}}{(c_1)_{j_1} \ldots (c_m)_{j_m} j_1! \ldots j_m!}, \tag{1.2}$$

where the series expansion is valid for all $z_i \in \mathbb{R}$. Using the results

$$(a)_j = \frac{\Gamma(a+j)}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_0^{\infty} \exp(-t) t^{a+j-1} dt,$$

for $j = 0, 1, 2, \ldots$, and $\sum_{j_i=0}^{\infty} \frac{(tz_i)^{j_i}}{(c_i)_{j_i} j_i!} = {}_0F_1(c_i; tz_i)$ in (1.2), we obtain

$$\Psi_2^{(m)}(a; c_1, \ldots, c_m; z_1, \ldots, z_m) = \frac{1}{\Gamma(a)} \int_0^{\infty} \exp(-t) t^{a-1} \prod_{i=1}^{m} {}_0F_1(c_i; tz_i) dt. \tag{1.3}$$

For further details of this function the reader is referred to Srivastava and Kashyap ([1], Section II.7) and Srivastava and Karlsson ([2], Section 1.4).

The non-central Dirichlet Type 2 distribution has many interesting properties (see [3,4]). In this paper, we give an extension to very definition and properties of the non-central Dirichlet Type 2 distribution on symmetric matrices, where the non-central Wishart distribution replaces the non-central gamma.

Let V_r be the space of real symmetric $r \times r$ matrices, Ω_r be the cone of positive definite and $\overline{\Omega_r}$ the cone of positive semi-definite elements of V_r. The identity matrix is denoted by I_r, the determinant of an element x of V_r by $\det(x)$ and its trace by $tr(x)$. We will use the notion of "quotient" defined by the division algorithm on matrices based on the Cholesky decomposition. More precisely, we use the fact that an element y of Ω_r can be written in a unique manner as $y = tt'$, where t is a lower triangular matrix with strictly positive diagonal and t' is its transpose. For an element x in V_r, we set $y(x) = txt'$, and the "quotient" of x by y is then defined as $y^{-1}(x) = t^{-1}xt'^{-1}$.

Now, we define Wishart and non-central Wishart distributions and state some of their properties. These definitions and results have been taken from ([5], Chapter 3).

A random matrix S is said to have Wishart distribution with parameters $p > \frac{r-1}{2}$ and $\Sigma \in \Omega_r$, denoted by $W_r(p, \Sigma)$, if its pdf is given by

$$\{\det(\Sigma)^p \Gamma_r(p)\}^{-1} \exp(-tr(s\Sigma^{-1})) \det(s)^{p-\frac{r+1}{2}} \mathbf{1}_{\Omega_r}(s),$$

where $\Gamma_r(.)$ is the multivariate gamma function defined by

$$\Gamma_r(p) = (2\pi)^{\frac{r(r-1)}{4}} \prod_{k=1}^{r} \Gamma\left(p - \frac{k-1}{2}\right).$$

A random matrix S is said to have a non-central Wishart distribution with parameters $p > \frac{r-1}{2}$, $\Sigma \in \Omega_r$ and $\Theta \in \overline{\Omega_r}$, denoted by $W_r(p, \Sigma, \Theta)$, if its pdf is given by

$$\{\det(\Sigma)^p \Gamma_r(p)\}^{-1} \exp(-tr(\Theta)) \exp(-tr(s\Sigma^{-1})) \det(s)^{p-\frac{r+1}{2}} {}_0F_1(p, \Theta\Sigma^{-1}s) \mathbf{1}_{\Omega_r}(s),$$

where ${}_0F_1$ is the Bessel function of matrix argument.

For $\Theta = 0$, the non-central Wishart distribution reduces to Wishart distribution.

Further, when $\Sigma = I_r$ and $\Theta = diag(\theta^2, 0, \ldots, 0)$, the pdf of S simplifies to

$$\{\det(\Sigma)^p \Gamma_r(p)\}^{-1} \exp(-(\theta^2 + tr(s))) \det(s)^{p-\frac{r+1}{2}} {}_0F_1(p, \theta^2 s_{11}) \mathbf{1}_{\Omega_r}(s = (s_{ij})), \tag{1.4}$$

where ${}_0F_1$ is the Bessel function of scalar argument.