J. Math. Anal. Appl. 396 (2012) 119–132

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Some notes concerning Riemannian metrics of Cheeger Gromoll type

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ARTICLE INFO

Article history:
Received 31 January 2012
Available online 15 June 2012
Submitted by H.R. Parks

Keywords:
Geodesic
Killing vector field
Paraholomorphic tensor field
Para-Kähler-Norden metric

ABSTRACT

Let (M, g) be an n-dimensional Riemannian manifold and TM its tangent bundle. The purpose of the present paper is three-fold. Firstly, to study paraholomorphy property of two Riemannian metrics g_a and $g_{a,b}$ of Cheeger Gromoll type depending on one parameter and two parameters by using compatible paracomplex structures J_a and $J_{a,b}$ on the tangent bundle TM. Secondly, to classify Killing vector fields on the tangent bundle TM equipped with the Riemannian metric $g_{a,b}$. Finally, to give a detailed description of geodesics on the tangent bundle TM with respect to the Riemannian metric $g_{a,b}$.

1. Introduction

The research in the topic of differential geometry of tangent bundles over Riemannian manifolds begun with Sasaki, who constructed, in the original paper [1] of 1958, a Riemannian metric ${}^{S}g$ on the tangent bundle TM of a Riemannian manifold (M, g), which depends closely on the base metric g. Although the Sasaki metric is naturally defined, it has been shown in many papers that the Sasaki metric presents a kind of rigidity. In [2], Kowalski proved that if the Sasaki metric ${}^{S}g$ is locally symmetric, then the base metric g is flat and hence ${}^{S}g$ is also flat. In [3], Musso and Tricerri have demonstrated an extreme rigidity of ${}^{S}g$ in the following sense: if $(TM, {}^{S}g)$ is of constant scalar curvature, then (M, g) is flat. Inspired by a paper of Cheeger and Gromoll, they also defined a new g-natural metric ${}^{CG}g$ on the tangent bundle TM, which they called the Cheeger Gromoll metric [4]. Sekizawa (see [5]) computed geometric objects related to this metric. Later, Gudmundson and Kappos, in [6,7], have completed these results and shown that the scalar curvature of the Cheeger Gromoll metric is never constant if the metric on the base manifold has constant sectional curvature. Furthermore, Abbassi and Sarih have proved that the tangent bundle TM with the Cheeger Gromoll metric is never a space of constant sectional curvature (see [8]). In [9], the first author and his collaborators studied the paraholomorphy property of the Sasaki and Cheeger Gromoll metrics by using compatible paracomplex stuctures on the tangent bundle and showed that the Cheeger Gromoll metric is never paraholomorphic with respect to the compatible paracomplex structure.

A more general metric is given by Anastasiei in [10] which generalizes both of the two metrics mentioned above in the following sense: it preserves the orthogonality of the two distributions, on the horizontal distribution it is the same as on the base manifold, and finally the Sasaki and the Cheeger Gromoll metric can be obtained as particular cases of this metric. A compatible almost complex structure is also introduced and the tangent bundle TM becomes a locally conformal almost Kählerian manifold. In [11], Munteanu studied another Riemannian metric on the tangent bundle TM of a Riemannian manifold M which generalizes the Sasaki metric and Cheeger Gromoll metric and a compatible almost complex structure which confers a structure of locally conformal almost K ählerian manifold to TM together with the metric. He found conditions under which the tangent bundle TM is almost Kählerian, locally conformal Kählerian or Kählerian

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0022-247X/$ – see front matter © 2012 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2012.06.011

when the tangent bundle TM has constant sectional curvature or constant scalar curvature. On the other hand, Oproiu and his collaborators constructed natural metrics on the tangent bundles of Riemannian manifolds which possess interesting geometric properties [12–15]. All the preceding metrics belong to the wide class of the so-called g-natural metrics on the tangent bundle, initially classified by Kowalski and Sekizawa [16] and fully characterized by Abbassi and Sarih [17–19] (see also [20] for other presentation of the basic result from [16] and for more details about the concept of naturality).

The work is organized as follows: In Section 2, some introductory materials concerning with the tangent bundle TM over an n-dimensional Riemannian manifold M are collected. In Section 3, we first introduce paraholomorphic Norden manifolds (or para-Kähler-Norden manifolds) and then investigate the paraholomorphy property of two Cheeger Gromoll type Riemannian metrics g_a and $g_{a,b}$ by using compatible paracomplex structures J_a and $J_{a,b}$ on the tangent bundle TM, respectively. In Section 4, the adapted frame which allows the tensor calculus to be efficiently done is inserted in the tangent bundle TM. Killing vector fields on $(TM, g_{a,b})$ are classified; that is, general forms of all Killing vector fields on $(TM, g_{a,b})$ are found. Also, it is shown that if $(TM, g_{a,b})$ is the tangent bundle with the Cheeger Gromoll type Riemannian metric $g_{a,b}$ of a Riemannian, compact and orientable manifold (M, g) with vanishing first and second Betti numbers, then the Lie algebras of Killing vector fields on (M, g) and on $(TM, g_{a,b})$ are isomorphic. In Section 5, we study relations between geodesics on the base manifold (M, g) and geodesics on the tangent bundle $(TM, g_{a,b})$ by means of the adapted frame.

2. Preliminaries

Let M be an n-dimensional Riemannian manifold. Throughout this paper, all manifolds, tensor fields and connections are always assumed to be differentiable of class C^∞. Also, we denote by $\Im^p_q(M)$ the set of all tensor fields of type (p, q) on M.

Basic formulas on tangent bundles: Let TM be the tangent bundle over an n-dimensional Riemannian manifold M, and π the natural projection $\pi : TM \to M$. Let the manifold M be covered by a system of coordinate neighborhoods (U, x^i), where (x^i), $i = 1, \dots, n$ is a local coordinate system defined in the neighborhood U. Let (y^i) be the Cartesian coordinates in each tangent space T_PM at $P \in M$ with respect to the natural base $\left\{ \frac{\partial}{\partial x^i}\Big|_P \right\}$, P being an arbitrary point in U whose coordinates are (x^i). Then we can introduce local coordinates (x^i, y^i) on open set $\pi^{-1}(U) \subset TM$. We call them induced coordinates on $\pi^{-1}(U)$ from (U, x^i). The projection π is represented by $(x^i, y^i) \to (x^i)$. The indices $I, J, \dots$ run from 1 to 2 n, the indices $\bar{i}, \bar{j}, \dots$ run from $n + 1$ to $2n$. Summation over repeated indices is always implied.

Let $X = X^i \frac{\partial}{\partial x^i}$ be the local expression in U of a vector field X on M. Then the vertical lift ${}^VX$, the horizontal lift ${}^HX$ and the complete lift ${}^CX$ of X are given, with respect to the induced coordinates, by

$$ {}^VX = X^i \partial_{\bar{i}}, \tag{2.1}$$

$$ {}^HX = X^i \partial_i - y^j \Gamma^i_{jk} X^k \partial_{\bar{i}}, \tag{2.2}$$

and

$$ {}^CX = X^i \partial_i + y^s \partial_s X^i \partial_{\bar{i}}, \tag{2.3}$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial y^i}$ and Γ^i_{jk} are the coefficients of the Levi-Civita connection ∇ of g.

In particular, we have the vertical spray ${}^Vu$ and the horizontal spray ${}^Hu$ on TM defined by

$$ {}^Vu = y^{iV}(\partial_i) = y^i \partial_{\bar{i}}, \qquad {}^Hu = y^{iH}(\partial_i) = y^i \delta_i, \tag{2.4}$$

where $\delta_i = \partial_i - y^j \Gamma^s_{ji} \partial_{\bar{s}}$. ${}^Vu$ is also called the canonical or Liouville vector field on TM.

Now, let r be the norm of a vector $u \in TM$. Then, for any smooth function f of R to R, we have

$$ {}^HX\left(f(r^2)\right) = 0 \tag{2.5}$$

$$ {}^VX\left(f(r^2)\right) = 2f'(r^2)g(X, u) \tag{2.6}$$

and in particular, we get

$$ {}^HX(r^2) = 0. \tag{2.7}$$

$$ {}^VX(r^2) = 2g(X, u). \tag{2.8}$$

Let X, Y and Z be any vector fields on M, then we have

$$ {}^HX(g(Y, u)) = g\left((\nabla_X Y), u\right), \tag{2.9}$$

$$ {}^VX(g(Y, u)) = g\left(X, Y\right), \tag{2.10}$$

$$ {}^HX\left({}^V(g(Y, Z))\right) = X\left(g(Y, Z)\right) \tag{2.11}$$

$$ {}^VX\left({}^V(g(Y, Z))\right) = 0 \text{ [19]}. \tag{2.12}$$