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Toeplitz operators on radially weighted harmonic Bergman spaces

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ABSTRACT

We study Toeplitz operators with uniformly continuous symbols on radially weighted harmonic Bergman spaces of the unit ball in $\mathbb{R}^n$. We describe their essential spectra and establish a short exact sequence associated with the C^*-algebra generated by these operators.

1. Introduction

Let $n \geq 2$ be a fixed integer. We write $\mathbb{B}$ for the open unit ball and $\mathbb{S}$ for the unit sphere in $\mathbb{R}^n$. The closure of $\mathbb{B}$, which is the closed unit ball, is denoted by $\bar{\mathbb{B}}$. For any $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, we use $|x|$ to denote the Euclidean norm of x, that is, $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$.

Let ν be a regular Borel probability measure on $\mathbb{B}$ that is invariant under the action of the group of orthogonal transformations $O(n)$. Then there is a regular Borel probability measure μ on the interval $[0, 1)$ so that the integration in polar coordinates formula

$$\int_{\mathbb{B}} f(x)\mathrm{d}\nu(x) = \int_{[0,1)} \int_{\mathbb{S}} f(r\zeta)\mathrm{d}\sigma(\zeta)\mathrm{d}\mu(r)$$

holds for all functions f that belong to $L^1(\mathbb{B}, \nu)$. Here σ is the unique $O(n)$-invariant regular Borel probability measure on the unit sphere $\mathbb{S}$. We are interested in measures ν whose support is not entirely contained in a compact subset of the unit ball so we will assume throughout the paper that $\nu(\{x \in \mathbb{B} : |x| \geq r\}) > 0$ for all $0 < r < 1$. This is equivalent to the condition that $\mu([r, 1)) > 0$ for all $0 < r < 1$.

The harmonic Bergman space b^2_ν is the space of all harmonic functions that belong also to the Hilbert space $L^2_\nu = L^2(\mathbb{B}, \nu)$. It follows from Poisson integral representation of harmonic functions and the assumption about ν that for any compact subset K of $\mathbb{B}$, there is a constant C_K such that

$$|u(x)| \leq C_K \|u\| = \left(\int_{\mathbb{B}} |u(x)|^2 \mathrm{d}\nu(x)\right)^{1/2} \tag{1.1}$$

for all x in K and all u in b^2_ν. This implies that b^2_ν is a closed subspace of L^2_ν and that the evaluation map $u \mapsto u(x)$ is a bounded linear functional on b^2_ν for each x in $\mathbb{B}$. By the Riesz representation, there is a function R_x in b^2_ν so that $u(x) = \langle u, R_x\rangle$. The function $R(y, x) := R_x(y)$ for $x, y \in \mathbb{B}$ is called the reproducing kernel for b^2_ν.

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Let Q denote the orthogonal projection from L^2_ν onto b^2_ν. For a bounded measurable function f on $\mathbb{B}$, the Toeplitz operator $T_f : b^2_\nu \to b^2_\nu$ is defined by

$$T_f u = QM_f u = Q(fu), \quad u \in b^2_\nu.$$

Here $M_f : L^2_\nu \to L^2_\nu$ is the operator of multiplication by f. The function f is called the symbol of T_f. We also define the Hankel operator $H_f : b^2_\nu \to (b^2_\nu)^\perp$ by

$$H_f u = (1 - Q)M_f u = (1 - Q)(fu), \quad u \in b^2_\nu.$$

It is immediate that $\|T_f\| \le \|f\|_\infty$ and $\|H_f\| \le \|f\|_\infty$.

For f, g bounded measurable functions on $\mathbb{B}$, the following basic properties are immediate from the definition of Toeplitz and Hankel operators:

$$T_{gf} - T_g T_f = H^*_{\bar{g}} H_f, \tag{1.2}$$

and

$$(T_g)^* = T_{\bar{g}}, \qquad T_{af+bg} = aT_f + bT_g,$$

where a, b are complex numbers and $\bar{g}$ denotes the complex conjugate of g.

If $\mathrm{d}\nu(x) = \mathrm{d}V(x)$, where V is the normalized Lebesgue volume measure on $\mathbb{B}$, then b^2_ν is the usual unweighted harmonic Bergman space. See [1, Chapter 8] for more details about this space. If $\mathrm{d}\nu(x) = c_\alpha(1 - |x|^2)^\alpha \mathrm{d}V(x)$ where $-1 < \alpha < \infty$ and c_α is a normalizing constant, then ν is a weighted Lebesgue measure on $\mathbb{B}$ and b^2_ν is a weighted harmonic Bergman space. Compactness of certain classes of Toeplitz operators on these weighted harmonic Bergman spaces was considered by Stroethoff in [2]. He also described the essential spectra of Toeplitz operators with uniformly continuous symbols. He showed that if f is a continuous function on the closed unit ball $\bar{\mathbb{B}}$, then the essential spectrum of T_f is the same as the set $f(\mathbb{S})$. This result in the setting of unweighted harmonic Bergman spaces was obtained earlier by Miao [3]. More recently, Choe et al. [4] have shown that the above essential spectral formula remains valid for unweighted harmonic Bergman spaces of any bounded domain with smooth boundary in $\mathbb{R}^n$. The common approach, which was used in all of the above papers, involved a careful estimate on the kernel function. In the case where ν is not a weighted Lebesgue measure, it seems that similar estimates are not available. Nevertheless, with a different approach, we still obtain the aforementioned essential spectral formula.

Let $\mathfrak{T}$ denote the C^*-algebra generated by all Toeplitz operators T_f, where f belongs to the space $C(\bar{\mathbb{B}})$ of continuous functions on the closed unit ball. Let $\mathfrak{CT}$ denote the two-sided ideal of $\mathfrak{T}$ generated by commutators $[T_f, T_g] = T_f T_g - T_g T_f$, for $f, g \in C(\bar{\mathbb{B}})$. For the case $n = 2$ and ν the normalized Lebesgue measure on the unit disk, Guo and Zheng [5] proved that $\mathfrak{CT} = \mathcal{K}$, the ideal of compact operators on b^2_ν, and there is a short exact sequence

$$0 \to \mathcal{K} \to \mathfrak{T} \to C(\mathbb{S}) \to 0.$$

They also proved that any Fredholm operator in the Toeplitz algebra $\mathfrak{T}$ has Fredholm index 0. We will show that these results are in fact valid for all $n \ge 2$.

The paper is organized as follows. In Section 2 we give some preliminaries. We then study Toeplitz operators with uniformly continuous symbols and establish the essential spectral formula in Section 3. The Toeplitz algebra and the associated short exact sequence are studied in Section 4. We close the paper with a criterion for compactness of operators with more general symbols in Section 5.

2. Preliminaries

It is well known that the reproducing kernel $R(x, y)$ is symmetric and real-valued for $x, y \in \mathbb{B}$. From (1.1) we see that for any compact subset K of $\mathbb{B}$ and $x \in K$,

$$R(x, x) = R_x(x) \le C_K \|R_x\| = C_K(\langle R_x, R_x\rangle)^{1/2} = C_K(R(x, x))^{1/2}. \tag{2.1}$$

This shows that $0 \le R(x, x) \le C_K^2$ for $x \in K$. So the function $x \mapsto K(x, x)$ is bounded on compact subsets of $\mathbb{B}$.

A polynomial p in the variable x with complex coefficients is homogeneous of degree m (or m-homogeneous), where $m \ge 0$ is an integer, if $p(tx) = t^m p(x)$ for all non-zero real numbers t. We write $\mathcal{P}_m$ for the vector space of all m-homogeneous polynomials on $\mathbb{R}^n$. We use $\mathcal{H}_m$ to denote the subspace of $\mathcal{P}_m$ consisting of harmonic polynomials. The subspace $\mathcal{H}_m$ is finite dimensional and its dimension h_m is given by $h_0 = 1$, $h_1 = n$ and $h_m = \binom{n+m-1}{n-1} - \binom{n+m-3}{n-1}$ for $m \ge 2$. See [1, Proposition 5.8].

For polynomials p in $\mathcal{H}_m$ and q in $\mathcal{H}_k$, using the orthogonality of their restrictions on the sphere [1, Proposition 5.9] and integration in polar coordinates we obtain

$$\begin{aligned} \langle p, q\rangle &= \left(\int_{[0,1)} r^{m+k} \mathrm{d}\mu(r)\right) \int_{\mathbb{S}} p\bar{q}\,\mathrm{d}\sigma \\ &= \begin{cases} 0 & \text{if } m \ne k \\ \left(\int_{[0,1)} r^{2m} \mathrm{d}\mu(r)\right) \int_{\mathbb{S}} p\bar{q}\,\mathrm{d}\sigma & \text{if } m = k. \end{cases} \end{aligned} \tag{2.2}$$