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Linearizability of homogeneous quartic polynomial systems with 1:−2 resonance

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ABSTRACT

In this paper, we consider the linearizability problem of complex planar polynomial systems of the form $\dot{x} = x + P_4$, $\dot{y} = -2y - Q_4$, where P_4 and Q_4 are homogeneous quartic polynomials. We obtain the necessary and sufficient conditions for linearizing the systems.

1. Introduction and main results

Recall the classical center problem (or Dulac's problem) for real planar vector fields

$$\dot{x} = \frac{dx}{dt} = -y + P(x, y), \qquad \dot{y} = \frac{dy}{dt} = x + Q(x, y), \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ are polynomials of degree n without constant and linear parts. One has to find conditions, on the coefficients of $P(x, y)$ and $Q(x, y)$, under which a neighborhood of the origin is covered by periodic solutions of the system. When system (1) has a center at the origin, if all the periodic solutions in the center area have the same period, the center is called isochronous center. According to a result of Poincaré and Lyapunov, the problem of isochronous centers is equivalent to the linearizability problem of system (1) (see Definition 2).

Let $x + iy = z \in \mathbb{C}$, $i^2 = -1$ and $t \to it$, then system (1) is transformed into a complex system

$$\dot{z} = z + \cdots, \qquad \dot{\bar{z}} = -\bar{z} + \cdots.$$

Thus, a real system with a center corresponds to a complex system

$$\dot{x} = x + \cdots, \qquad \dot{y} = -y + \cdots,$$

with a 1:−1 resonant saddle and a local analytic first integral $H = xy + h \cdot o \cdot t$ at the origin (in our paper, $h \cdot o \cdot t$ expresses higher order terms). Therefore a natural generalization of the center problem is to consider a polynomial complex planar vector field of the form

$$\dot{x} = px + P(x, y), \qquad \dot{y} = -qy - Q(x, y), \tag{2}$$

where P and Q are complex polynomials and p and q are positive integers with $(p, q) = 1$. Such a generalization has been considered in [1–3]. The corresponding center problem becomes the integrability problem of system (2).

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The linearizability problem is one of particularly important cases in the center problem and also has a significant application in qualitative analysis and bifurcation theory. Clearly, if a system is linearizable, then it is integrable, but not vice versa.

The linearizability problem has been extensively studied by many authors. For example, when $p = q = 1$, in [4,5], the authors have completely solved the cases with both P and Q being homogeneous polynomials of degrees 2, 3 and 5. Remark that, in real case, the center problem has also been studied for quadratic P and Q (Bautin [6]) and for homogeneous cubic P and Q (Sibirskii [7]). In [8,9], the necessary and sufficient conditions of linearizability for time-reversible cubic and homogeneous quartic systems are given. For $p = 1$ and $q = 2$, the quadratic system is studied in [2]. More studies can be found in, e.g., [1,4,10–14]. For general p and q, even for quadratic systems, this problem is still open. In fact, even for the Lotka–Volterra system, it is far from being solved.

Historically, the progress in solving the linearizability problem is pretty slow. The main reason is that the unique way to obtain necessary conditions of linearizability is to calculate the so-called linearizability quantities (see Definition 4) which are polynomials in the coefficients of system (2). These quantities can be computed algorithmically. However, with the growth of the order, usually they become too complicated to handle.

In this paper, we shall mainly discuss the linearizability problem of a class of system (2). In other words, we shall look for the necessary and sufficient conditions in terms of the coefficients of P and Q. More precisely, we consider the following system

$$\dot{x} = x + P_4(x, y), \qquad \dot{y} = -2y - Q_4(x, y), \tag{3}$$

in $\mathbb{C}^2$, where P_4 and Q_4 are homogeneous polynomials of degree 4. In terms of a given coordinate system, we denote

$$\begin{aligned} P_4 &= a_4x^4 + a_3x^3y + a_2x^2y^2 + a_1xy^3 + a_0y^4, \\ Q_4 &= b_0x^4 + b_1x^3y + b_2x^2y^2 + b_3xy^3 + b_4y^4. \end{aligned} \tag{4}$$

Notice that if $a_0b_0 \neq 0$, they can be reduced to any given nonzero constants by a linear scaling. Clearly, system (3) is still linearizable after a linear scaling. In this sense, the main results of the paper is the following theorem.

Main Theorem. *System (3) is linearizable if and only if $a_3 = b_2 = 0$ and one of the following conditions holds:*

(I) $a_0 = a_1 = a_2 = b_3 = b_4 = 0$;
(II) $a_4 = b_0 = b_1 = 0$;
(III) $a_0 = a_1 = a_2 = b_0 = b_3 = b_1 - a_4 = 0$;
(IV) $a_0 = a_2 = b_0 = b_1 = b_3 = 0$;
(V) $a_0 = b_0 = b_1 + a_4 = b_3 + a_2 = b_4 + a_1 = 0$;
(VI) $a_0 - 1 = a_1 = a_2 = b_0 = b_3 = b_4 = 4b_1 + a_4 = 0$;
(VII) $a_1 = a_2 = b_0 = b_4 = 2b_1 - a_4 = 0$;
(VIII) $a_0 = a_1 = a_4 = b_0 - 1 = b_1 = b_4 = b_3 + 4a_2 = 0$;
(IX) $a_0 = b_0 - 1 = b_1 - 1 = a_1 - 864 = a_2 - 108 = a_4 + 7 = b_3 + 324 = b_4 - 432 = 0$;
(X) $a_0 - 9 = a_1 + 8 = a_2 + 6 = a_4 - 5 = b_0 + 3 = b_1 + 2 = b_3 - 18 = b_4 + 13 = 0$.

We present the corresponding systems as follows. They are more convenient forms for judging whether a given system of the form (3) is linearizable or not.

(I) $\dot{x} = x + a_4x^4, \dot{y} = -2y - b_0x^4 - b_1x^3y$;
(II) $\dot{x} = x + a_2x^2y^2 + a_1xy^3 + a_0y^4, \dot{y} = -2y - b_3xy^3 - b_4y^4$;
(III) $\dot{x} = x + a_4x^4, \dot{y} = -2y - a_4x^3y - b_4y^4$;
(IV) $\dot{x} = x + a_4x^4 + a_1xy^3, \dot{y} = -2y - b_4y^4$;
(V) $\dot{x} = x(1 + a_4x^3 + a_2xy^2 + a_1y^3), \dot{y} = y(-2 + a_4x^3 + a_2xy^2 + a_1y^3)$;
(VI) $\dot{x} = x + a_4x^4 + y^4, \dot{y} = -2y + \frac{a_4}{4}x^3y$;
(VII) $\dot{x} = x + a_4x^4 + a_0y^4, \dot{y} = -2y - \frac{a_4}{2}x^3y - b_3xy^3$;
(VIII) $\dot{x} = x + a_2x^2y^2, \dot{y} = -2y - x^4 + 4a_2xy^3$;
(IX) $\dot{x} = x - 7x^4 + 108x^2y^2 + 864xy^3, \dot{y} = -2y - x^4 - x^3y + 324xy^3 - 432y^4$;
(X) $\dot{x} = x + (x - y)^2(5x^2 + 10xy + 9y^2), \dot{y} = -2y + (x - y)^2(3x^2 + 8xy + 13y^2)$.

We noticed that very long lists of classification in the 1:−1 homogeneous time-reversible quartic case and the homogeneous quintic case studied in [8,5] can be essentially rearranged and put into much more compact forms like given here.

The proof of Main Theorem will be given in Section 3. In what follows, we first introduce some preliminaries about general methods for linearizability.