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Multiplicity of positive solutions for a critical quasilinear elliptic system with concave and convex nonlinearities

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ABSTRACT

In this paper, by using the Lusternik–Schnirelmann category, we obtain a multiplicity result for a quasilinear elliptic system with both concave and convex nonlinearities and critical growth terms in bounded domains.

1. Introduction

In this paper, we are concerned with the multiplicity of positive solutions of the following p-Laplacian elliptic system:

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u + \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ -\Delta_p v = \mu |v|^{q-2}v + \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $0 \in \Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $N \geq 3$, $\lambda, \mu > 0$ are parameters and $\alpha, \beta > 1$ satisfying $\alpha + \beta = p^*$ ($p^* := \frac{pN}{N-p}$, $p < N$, denotes the critical Sobolev exponent). We assume that $1 < q < p$. Recently, Hsu in [1] has proved the existence of at least two positive solutions of problem (1) if the pair of the parameters (λ, μ) belongs to a certain subset of $\mathbb{R}^2$. Our purpose here is to reley the number of positive solutions of problem (1) to the topology of Ω. The main result is the following.

Theorem 1. *Assume that $N > p^2$ and $p^* - \frac{N}{N-p} \leq q < p$. Then, there exists $\Lambda_* > 0$ such that for each $\lambda, \mu \in (0, \Lambda_*)$, problem (1) has at least $cat(\Omega) + 1$ distinct positive solutions.*

When $q \geq p$, employing the Lusternik–Schnirelmann category, it was shown in [2] that if $N \geq p^2$ and $2 \leq p \leq q \leq p^*$, then (1) has at least $cat(\Omega)$ distinct solutions for $\lambda, \mu > 0$ small enough. For more similar results, we refer the reader to [3–5]. When $q < p$, in striking contrast, very few is known about the eventual role of the topology of the domain on the multiplicity question of system (1). To establish our main result we follow, as in [3,2], a classical approach and borrow some

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techniques of [6] and arguments developed in [7]. This paper is organized as follows. In Section 2, we fix some notations and give some preliminary results and known facts. In Section 3, we show some technical lemmas which enable us to construct homotopies between Ω and certain sublevel set of the energy functional associated to (1). In Section 4, we prove Theorem 1.

2. Some notations and preliminaries

The space X designates the Sobolev space $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ equipped by its usual norm

$$\|(u, v)\| := \left(\int_\Omega (|\nabla u|^p + |\nabla v|^p)dx\right)^{\frac{1}{p}}.$$

Solutions to problem (1) will be obtained as critical points of the corresponding energy functional

$$I_{\lambda,\mu}(u, v) := \frac{1}{p}\int_\Omega (|\nabla u|^p + |\nabla v|^p)dx - \frac{1}{q}K_{\lambda,\mu}(u^+, v^+) - \frac{2}{\alpha+\beta}R(u^+, v^+), \tag{2}$$

where

$$K_{\lambda,\mu}(u, v) := \int_\Omega (\lambda|u|^q + \mu|v|^q)dx$$

and

$$R(u, v) := \int_\Omega |u|^\alpha |v|^\beta dx.$$

$N_{\lambda,\mu}$ denotes the Nehari manifold related to $I_{\lambda,\mu}$, given by

$$N_{\lambda,\mu} := \{(u, v) \in X \setminus \{0\} : I'_{\lambda,\mu}(u, v)(u, v) = 0\}.$$

We consider the modified functional $\widetilde{I}_{\lambda,\mu}$ defined on $\mathbb{R} \times X$ by

$$\widetilde{I}_{\lambda,\mu}(t, u, v) := I_{\lambda,\mu}(tu, tv).$$

In the sequel, we shall use Λ_* to denote different small parameters.

Lemma 2. *There exists $\Lambda_* > 0$ such that if $\lambda, \mu \in (0, \Lambda_*)$ and for every $(u, v) \in X \setminus \{0\}$ with $(u^+, v^+) \neq (0, 0)$, the real valued function $t \mapsto \partial_t \widetilde{I}_{\lambda,\mu}(t, u, v)$ has a unique positive zero denoted by $t_1(u, v, \lambda, \mu)$ such that*

$$\partial_{tt}\widetilde{I}_{\lambda,\mu}(t_1(u, v, \lambda, \mu), u, v) > 0.$$

Moreover if $R(u^+, v^+) > 0$, the real valued function $t \mapsto \partial_t \widetilde{I}_{\lambda,\mu}(t, u, v)$ has exactly two positive zeros denoted by $t_1(u, v, \lambda, \mu)$ and $t_2(u, v, \lambda, \mu)$ with

$$\partial_{tt}\widetilde{I}_{\lambda,\mu}(t_1(u, v, \lambda, \mu), u, v) > 0 \quad \text{and} \quad \partial_{tt}\widetilde{I}_{\lambda,\mu}(t_2(u, v, \lambda, \mu), u, v) < 0.$$

In particular, we have

$$t_1(u, v, \lambda, \mu) < t(u, v) < t_2(u, v, \lambda, \mu),$$
$$I_{\lambda,\mu}(t_1(u, v, \lambda, \mu)u, t_1(u, v, \lambda, \mu)v) = \min_{0 \le t \le t(u,v)} I_{\lambda,\mu}(tu, tv)$$

and

$$I_{\lambda,\mu}(t_2(u, v, \lambda, \mu)u, t_2(u, v, \lambda, \mu)v) = \max_{t \ge 0} I_{\lambda,\mu}(tu, tv),$$

where

$$t(u, v) := \left[\frac{p-q}{2(p^*-q)} \frac{\int_\Omega (|\nabla u|^p + |\nabla v|^p)dx}{R(u^+, v^+)}\right]^{\frac{1}{p^*-p}}.$$

Proof. The proof is almost the same as that in Brown–Wu [8, Lemma 2.6] (or see Tarantello [9, Lemma 1]) and is omitted here. □

Remark 1. We observe from Lemma 2 that we can split $N_{\lambda,\mu}$ into two disjoint parts:

$$N^+_{\lambda,\mu} := \{(t_1(u, v, \lambda, \mu)u, t_1(u, v, \lambda, \mu)v) : (u, v), (u^+, v^+) \in X \setminus \{0\}\},$$
$$N^-_{\lambda,\mu} := \{(t_2(u, v, \lambda, \mu)u, t_2(u, v, \lambda, \mu)v) : (u, v) \in X \setminus \{0\}, R(u^+, v^+) > 0\}.$$

We denote by $c_{\lambda,\mu}$ the following number:

$$c_{\lambda,\mu} := \inf\{I_{\lambda,\mu}(t_2(u, v, \lambda, \mu)u, t_2(u, v, \lambda, \mu)v) : (u, v) \in X, R(u^+, v^+) > 0\}.$$