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p-adic (2, 1)-rational dynamical systems

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ABSTRACT

We study the dynamics of an arbitrary (2, 1)-rational function $f(x) = (x^2+ax+b)/(cx+d)$ on the field $\mathbb{C}_p$ of complex p-adic numbers. We show that the p-adic dynamical system generated by f has a very rich behavior. Siegel disks may either coincide or be disjoint for different fixed points of the dynamical system. Also, we find the basin of the attractor of the system. Varying the parameters, it is proven that there are periodic trajectories. For some values of the parameters there are trajectories which go arbitrary far from fixed points.

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1. Introduction

In this paper we will state some results concerning discrete dynamical systems defined over the complex p-adic field $\mathbb{C}_p$. The interest in such systems and in the ways in which they can be applied has been rapidly increasing during the last couple of decades (see, e.g., [1] and the references therein). The p-adic numbers were first introduced by the German mathematician Hensel. For about a century after the discovery of p-adic numbers, they were mainly considered objects of pure mathematics. Beginning in the 1980s, various models described in the language of p-adic analysis have been actively studied. The study of models over the field of p-adic numbers was based on the conception that p-adic numbers might provide a more exact and more adequate description of micro-world phenomena. Numerous applications of these numbers in theoretical physics have been proposed in the papers [2–6] to quantum mechanics [7], to p-adic-valued physical observables [7] and in many other areas [8,9].

The study of p-adic dynamical systems arises in Diophantine geometry in the constructions of canonical heights, used for counting rational points on algebraic varieties over a number field, as in [10].

The most recent monograph on p-adic dynamics is Anashin and Khrennikov [11]; nearly half of Silverman's monograph [12] also concerns p-adic dynamics.[1]

There are many areas where p-adic dynamics proved to be effective: computer science (straight line programs), numerical analysis and simulations (pseudorandom numbers), uniform distribution of sequences, cryptography (stream ciphers, T-functions), combinatorics (Latin squares), automata theory and formal languages, and genetics. The monograph [11] contains a corresponding survey. For newer results see some recent papers and references therein: [13–21]. Moreover, there are studies in computer science and cryptography which, along with mathematical physics, stimulated in the 1990s intensive research into p-adic dynamics since it was observed that major computer instructions (and therefore programs composed of these instructions) can be considered as continuous transformations with respect to the 2-adic metric; see [22,23].

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In [24,25] p-adic fields have arisen in physics in the theory of superstrings, giving rise to questions about their dynamics. Also some applications of p-adic dynamical systems to some biological and physical systems have been proposed in [26–30, 24,31]. Other studies of non-Archimedean dynamics in the neighborhood of a periodic point and of the counting of periodic points over global fields using local fields appear in [32–34]. It is known that the analytic functions play important roles in complex analysis. In the p-adic analysis the rational functions play a role similar to that of analytic functions in complex analysis [35]. Therefore, there naturally arises a question as regards the study the dynamics of these functions in p-adic analysis. On the other hand, these p-adic dynamical systems appear when studying p-adic Gibbs measures [36–41]. In [42,43], dynamics on the Fatou set of a rational function defined over some finite extension of $\mathbb{Q}_p$ have been studied, besides which an analogue of Sullivan's no wandering domains theorem for p-adic rational functions which have no wild recurrent Julia critical points was proved. In [27,44] the behavior and ergodicity of a p-adic dynamical system $f(x) = x^n$ in the fields of p-adic numbers $\mathbb{Q}_p$ and complex p-adic numbers $\mathbb{C}_p$ were investigated. Firstly, the problem of ergodicity of perturbed monomial dynamical systems, which was posed in these papers and which stimulated intensive research, was solved in [45]. Secondly, quite recently a far-reaching generalization of the problem for arbitrary 1-Lipschitz transformations of 2-adic spheres was also solved, in [46]. Finally, we note that it is not only polynomial and rational p-adic dynamical systems that have been studied: in the past decade, significant progress has been achieved in the study of very general p-adic dynamical systems: non-expansive, locally analytic, shift-like, etc. The basics of p-adic analysis and of p-adic mathematical physics are explained in [47,48,9].

In [49], the behavior of a trajectory of a rational p-adic dynamical system in a complex p-adic field $\mathbb{C}_p$ is studied. The paper studies Siegel disks and attractors of those dynamical systems. It is shown that Siegel disks may either coincide or be disjoint for different fixed points. Also, the basin of the attractor of the rational dynamical system is found.

In this paper we consider the dynamical system associated with the $(2, 1)$-rational function $f : \mathbb{C}_p \to \mathbb{C}_p$ defined by

$$f(x) = \frac{x^2 + ax + b}{cx + d}, \quad a, b, c, d \in \mathbb{C}_p,\ c \neq 0,\ d^2 - acd + bc^2 \neq 0,$$

where $x \neq \hat{x} = -\frac{d}{c}$. The paper is organized as follows. In Section 2 we give some preliminaries. Section 3 contains the definition of a $(2, 1)$-rational function. Section 4 is devoted to the p-adic dynamical system which has a unique fixed point x_0. We prove that if the point is attracting then there exists $\delta > 0$ such that the basin of attraction for x_0 is the ball of radius δ and centered at x_0, and any sphere of radius $\geq \delta$ is invariant. If x_0 is an indifferent point, then all spheres centered at x_0 are invariant. If x_0 is a repelling point, then there exists $\delta > 0$ such that the trajectory which starts at an element of the ball of radius δ with center at x_0 leaves this ball, whereas any sphere of radius $\geq \delta$ is invariant. Section 5 contains results concerning p-adic dynamical systems which have no fixed point. We show that these p-adic dynamical systems have a 2-periodic cycle x_1, x_2 which can only be an attracting or an indifferent one. If it is attracting then it attracts each trajectory which starts from an element of a ball of radius $r = |x_1 - x_2|_p$ centered at x_1 or at x_2. If the 2-periodic cycle is an indifferent one then every iteration maps either of the two aforementioned balls to another one. All other spheres of radius $> r$ and center x_1 and x_2 are invariant independently of the attractiveness of the cycle. The last section is devoted to the case when there are two distinct fixed points. We show, in particular, that Siegel disks may either coincide or be disjoint for different fixed points of the dynamical system. Also, we find the basin of the attractor of the system. Varying the parameters, we prove that there exists $k \geq 2$ and spheres $S_{r_1}(x_i), \dots, S_{r_k}(x_i)$ such that the limiting trajectory will be periodically traveling to the spheres $S_{r_j}(x_i)$. For some values of the parameters there are trajectories which go arbitrary far from the fixed points.

2. Preliminaries

2.1. p-adic numbers

Let $\mathbb{Q}$ be the field of rational numbers. The greatest common divisor of the positive integers n and m is denoted by (n, m). Every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in \mathbb{Z}$, m is a positive integer, $(p, n) = 1$, $(p, m) = 1$ and p is a fixed prime number.

The p-adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

It has the following properties:

(1) $|x|_p \geq 0$ and $|x|_p = 0$ if and only if $x = 0$,
(2) $|xy|_p = |x|_p |y|_p$,
(3) the strong triangle inequality holds:

$$|x + y|_p \leq \max\{|x|_p, |y|_p\},$$

(3.1) if $|x|_p \neq |y|_p$ then $|x + y|_p = \max\{|x|_p, |y|_p\}$,
(3.2) if $|x|_p = |y|_p$ then $|x + y|_p \leq |x|_p$.

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