J. Math. Anal. Appl. 398 (2013) 638–648

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

On set-valued iteration groups generated by commuting functions

Marek Cezary Zdun

Institute of Mathematics, Pedagogical University, ul. Podchorążych 2, PL-30-084 Kraków, Poland

ARTICLE INFO

Article history:
Received 23 March 2012
Available online 19 September 2012
Submitted by Michael J. Schlosser

Keywords:
Iteration groups
Multi-valued functions
Functional equations
Set-valued iteration semigroups
Embedding

ABSTRACT

Let I be an open interval and $f, g : I \to I$ be commuting homeomorphisms such that $f < id$, $g < id$ and $\inf\{\frac{n}{m} : n, m \in \mathbb{N}, f^n < g^m\} =: s \notin \mathbb{Q}$. Define $f_+^t := \sup\{f^n \circ g^{-m} : n, m \in \mathbb{N}, n - sm < t\}$, $f_-^t := \inf\{f^n \circ g^{-m} : n, m \in \mathbb{N}, n - sm > t\}$ and $F^t(x) := [f_-^t(x), f_+^t(x)]$, $x \in I$. We show that the family $\{F^t : I \to cc(I), t \in \mathbb{R}\}$ is an iteration group in the sense of set-valued functions such that $f(x) \in F^1(x)$ and $g(x) \in F^s(x)$. If f and g are embeddable in an iteration group of continuous functions $\{f^t : t \in \mathbb{R}\}$, then there exists an additive function γ such that $f^t(x) \in F^{\gamma(t)}(x)$, $t \in \mathbb{R}$. Several other properties of set-valued iteration groups $\{F^t : t \in \mathbb{R}\}$ as well as iteration groups $\{f_-^t : t \in \mathbb{R}\}$ and $\{f_+^t : t \in \mathbb{R}\}$ are given.

© 2012 Elsevier Inc. All rights reserved.

1. Introduction

In this note we consider the problem of the embeddability of commuting homeomorphisms of an open interval in multi-valued iteration groups. The problem arises when the commuting homeomorphisms are not the elements of any common iteration group. To solve the problem we consider the set of phantom iterates which extends the set of the original objects of iteration. The phantoms here are set-valued mappings. The idea of generalized embeddings appeared already in papers of Reich and Schweiger (see [1,2]) and has been used to find a necessary and sufficient conditions for the embeddability of formally biholomorphic mappings. Here the phantom iterates belong to groups of finite and infinite matrixes. This idea has been extended and popularized by Targonski (see [3–5]) but he considered another approach. He defined the phantom iterates using the Koopman operator and semigroups of linear operators on suitable commutative algebras. The notion of phantom iterates is not precisely defined. What is taken as a set of phantoms depends on the specific situation. In our case this is a set of set-valued functions where the values are the compact intervals.

In this note we use the notion of the set-valued iteration semigroups. These semigroups were introduced and investigated by Smajdor in [6], then studied by Smajdor and Olko in the case of semigroups of linear functions with compact values in Banach space (see e.g. [7,8]). Moreover, Łydzińska considered iteration semigroups of functions whose values are the arbitrary subsets of a given set (see [9,10]) and also in topological spaces (see [11]). She considered the set-valued iteration semigroups which are the counterparts of the fundamental form of continuous iteration semigroups for single-valued functions on an interval.

To solve our problem first we define for given commuting iteratively incommensurable homeomorphisms f and g of an open interval I two special iteration groups $\{f_-^t : I \to I, t \in \mathbb{R}\}$, $\{f_+^t : I \to I, t \in \mathbb{R}\}$ of non-decreasing functions such that $f_-^t \le f_+^t$ and then we use them for the construction of a special interval-valued iteration group $\{F^t : I \to cc[I], t \in \mathbb{R}\}$ where $F^t(x) = [f_-^t(x), f_+^t(x)]$. This phantom iteration group has a property that mappings f and g are the selections of suitable set-valued iterates F^u and F^v. In the first part of the paper we deal with iteration groups $\{f_\pm^t : I \to I, t \in \mathbb{R}\}$ and in the second part we show several important properties of the set-valued iteration group $\{F^t : I \to cc[I], t \in \mathbb{R}\}$.

There are several meanings of the notion of iteration groups. We shall use the following one. Let I be an interval.

E-mail address: mczdun@ap.krakow.pl.

0022-247X/$ – see front matter © 2012 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2012.09.016

Definition 1. A family $\{f^t: I \to I, t \in \mathbb{R}\}$ of mappings is said to be an iteration group if

$$f^t \circ f^s = f^{t+s} \quad t, s \in \mathbb{R}.$$

Usually f^0 is not the identity function.

Definition 2. If in an iteration group $\{f^t: I \to I, t \in \mathbb{R}\}$ every function f^t is continuous and for every $x \in I$ the mapping $t \mapsto f^t(x)$ is continuous, then the iteration group is said to be continuous.

If at least one mapping from a continuous iteration group $\{f^t: I \to I, t \in \mathbb{R}\}$ is a injection or a surjection then $f^0 = id$ and all iterates f^t are homeomorphisms.

Assume the following general hypothesis

(H) $f, g: I \to I$ are continuous bijections, $f(x) < x, g(x) < x$ for $x \in I$ and $f \circ g = g \circ f$.

Definition 3. The mappings f and g are said to be embeddable if there exists an iteration group of homeomorphisms $\{f^t: I \to I, t \in \mathbb{R}\}$ such that $f = f^1$ and $g = f^s$ for some $s \in \mathbb{R}$.

It is obvious that every two functions belonging to an iteration group commute, but it does not have to be true conversely. For every homeomorphism $f: I \to I$ one can find a homeomorphism g commuting with f such that they are not embeddable in any iteration group of homeomorphisms (see [12]). In [13] is also given a necessary and a sufficient condition when the functions f and g satisfying (H) are embeddable. In the present paper we concentrate on the case when f and g are not embeddable. For these functions we construct the special discontinuous iteration groups $\{f^t_\pm : I \to I, t \in \mathbb{R}\}$, where $f^t_\pm$ are monotonic discontinuous and nonsingular mappings with property that $f^1_- \le f \le f^1_+$ and $f^s_- \le g \le f^s_+$ for a $s > 0$. We prove that the family of set-valued functions $F^t(x) = [f^t_-(x), f^t_+(x)]$ for $t \in \mathbb{R}$ form a generalized iteration group such that f and g are respectively the selectors of the set-valued functions F^1 and F^s. On the one hand this set-valued iteration group has several regular properties and on the other hand each irregular iteration group of homeomorphisms containing f and g, if it exists, is a selection of a subgroup of the group $\{F^t : I \to cc[I], t \in \mathbb{R}\}$. Thus this set-valued group play a role of some kind of universe for nonmeasurable iteration groups of homeomorphisms embedding given commuting homeomorphisms.

2. Auxiliary facts

Let us note that if f and g satisfy (H) then for every $x \in I$ the sequence $\{f^n(x)\}$ is strictly decreasing and $\lim_{n\to\infty} f^n(x) = \inf I$. Moreover, for every $x \in I$ and every positive integer k there exists a unique nonnegative integer $m_k(x)$ such that

$$f^{m_k(x)+1}(x) < g^k(x) \le f^{m_k(x)}(x).$$

Proposition 4 (*[14]*). *If f and g satisfy* (H) *then for every $x \in I$ there exists a finite limit*

$$\lim_{k\to\infty} \frac{m_k(x)}{k} =: s(f, g)$$

and it does not depend on x. Moreover, $s(f, g) \notin \mathbb{Q}$ iff f and g are iteratively incommensurable, i.e. for every $x \in I$ and every $m, n \in \mathbb{N}$ such that $|n| + |m| > 0, f^n(x) \neq g^m(x)$.

In [15] it is shown that for every $x \in I$

$$s(f, g) = \inf\left\{\frac{m}{n} : m, n \in \mathbb{N}, f^n(x) < g^m(x)\right\}.$$

Proposition 5 (*[12]*). *Let f and g satisfy* (H) *and $s(f, g) \notin \mathbb{Q}$. Then the set*

$$L(f, g) := \{f^n \circ g^{-m}(x), n, m \in \mathbb{N}\}^d$$

does not depend on the choice of $x \in I$ and $L(f, g)$ is either a Cantor set or $L(f, g) = [a, b]$. Moreover, $f(L(f, g) \cap I) = L(f, g) \cup I$ and $g(L(f, g) \cap I) = L(f, g) \cup I$.

Note that if $\operatorname{Int} L(f, g) = \emptyset$ that is $L(f, g)$ is a Cantor set, then we have the following decomposition $I \setminus L(f, g) = \bigcup_{\alpha\in\mathbb{Q}} I_\alpha$, where and I_α, $\alpha \in \mathbb{Q}$ are open pairwise disjoint intervals. We can assume that if $\alpha < \beta$, then for every $u \in I_\alpha$ and $v \in I_\beta u < v$.

Proposition 6 (*[16]*). *Let f and g satisfy* (H) *and $s(f, g) \notin \mathbb{Q}$. Then the system of Abel equations*

$$\begin{cases} \varphi(f(x)) = \varphi(x) + 1 \\ \varphi(g(x)) = \varphi(x) + s(f, g), \end{cases} \quad x \in I \tag{1}$$

has a unique continuous solution φ up to an additive constant. This solution is decreasing. Moreover, the solution φ is invertible iff $\operatorname{Int} L(f, g) \neq \emptyset$.

If $\operatorname{Int} L(f, g) = \emptyset$, then the closure of each component of the set $I \setminus L(f, g)$ is a maximal interval of constancy of φ.

The form of continuous solution of (1) is described in paper [17]. Further we shall use a shorter notation $L := L(f, g)$ and $s := s(f, g)$.

Download English Version:

https://daneshyari.com/en/article/4617085

Download Persian Version:

https://daneshyari.com/article/4617085

Daneshyari.com