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Multiple solutions of sublinear elliptic equations with small perturbations

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ABSTRACT

We study the sublinear elliptic equation having two nonlinear terms, where the main term $f(x, u)$ is sublinear and odd with respect to u and the perturbation term is any continuous function with a small coefficient. Then we prove the existence of multiple small solutions.

1. Introduction and main results

In this paper we prove the existence of multiple solutions for the sublinear elliptic equation

$$\begin{cases} -\Delta u = f(x, u) + \varepsilon g(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ and ε is a small parameter. We study the problem (1.1) under the condition that $f(x, u)$ is odd on u, sublinear near $u = 0$ and $g(x, u)$ is any continuous function. Then we shall show that if $|\varepsilon|$ is small enough, (1.1) has many small solutions. We impose the next assumption.

Assumption (A). Let $f(x, u)$ and $g(x, u)$ be Hölder continuous functions defined on $\overline{\Omega} \times [-a, a]$ with some $a > 0$ and satisfy the conditions below.

(A1) $f(x, -u) = -f(x, u)$ for $x \in \overline{\Omega}$ and $|u| \le a$.
(A2) $uf(x, u) - 2F(x, u) < 0$ when $0 < |u| < a$ and $x \in \overline{\Omega}$. Here $F(x, u)$ is defined by

$$F(x, u) := \int_0^u f(x, s)ds.$$

(A3) $\lim_{u\to 0} \left(\min_{x\in\overline{\Omega}} u^{-2}F(x, u)\right) = \infty$.

Theorem 1.1. *Suppose that Assumption (A) holds. Then for any $k \in \mathbb{N}$ and any $\delta > 0$, there exists an $\varepsilon(k, \delta) > 0$ such that if $|\varepsilon| \le \varepsilon(k, \delta)$, then (1.1) has at least k distinct solutions whose $C^2(\overline{\Omega})$-norms are less than δ. When $\varepsilon = 0$, (1.1) has a sequence of solutions whose $C^2(\overline{\Omega})$-norm converges to zero.*

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Example 1.2. We give some examples of $f(x, u)$ which satisfy Assumption (A). In the following, we suppose that $\alpha(x)$ and $\beta(x)$ are Hölder continuous and $\alpha(x) > 0$ on $\overline{\Omega}$.

(i) $f(x, u) = \alpha(x)|u|^p \operatorname{sgn} u$ with $0 < p < 1$.
(ii) $f(x, u) = -\alpha(x)u \log |u|$.
(iii) $f(x, u) = \alpha(x)|u|^p \operatorname{sgn} u + \beta(x)|u|^q \operatorname{sgn} u$ with $0 < p < \min(1, q)$.

In Case (iii), $\beta(x)$ may change its sign. Indeed, we have

$$uf(x, u) - 2F(x, u) = -\frac{1-p}{1+p}\alpha(x)|u|^{p+1} - \frac{1-q}{1+q}\beta(x)|u|^{q+1} < 0,$$

provided that $|u| > 0$ is small enough. For these nonlinear terms, (1.1) has sufficiently many small solutions if $|\varepsilon|$ is small enough.

For the sublinear elliptic problem with $\varepsilon = 0$, i.e., $f(u)$ is like $|u|^p \operatorname{sgn} u$ with $0 < p < 1$, we refer the readers to [1,2,5]. Ambrosetti–Badiale [1] has proved the existence of infinitely many solutions if $f(x, u)$ is sublinear with $\varepsilon = 0$. Ambrosetti et al. [2] has investigated $f(u) = \lambda|u|^q \operatorname{sgn} u + |u|^p \operatorname{sgn} u$ with $0 < q < 1 < p \le (n+2)/(n-2)$. Then they have obtained the detailed and important results on the structure of positive solutions, the existence of two positive solutions and the existence of infinitely many solutions. Under more general and weak assumptions on $f(x, u)$, we have proved in [5] that (1.1) has a sequence of solutions whose $C^2(\overline{\Omega})$-norm converges to zero.

On the other hand, we have considered the sublinear perturbation problem in our paper [6] under the condition that $f(u) = |u|^p \operatorname{sgn} u$ with $0 < p < 1$, $\varepsilon = 1$, $g(x, 0) = 0$, $g(x, u)$ is not odd on u and $g(x, u)$ converges rapidly to zero as $u \to 0$. Then we have obtained a sequence of solutions whose $C^2(\overline{\Omega})$-norm converges to zero.

Degiovanni and Rădulescu [3] have proved the existence of multiple solutions $(u, \lambda) \in H_0^1(\Omega) \times \mathbb{R}$ of the problem

$$-\Delta u = \lambda(f(x, u) + g(x, u)) \quad \text{in } \mathcal{D}'(\Omega),$$
$$\int_\Omega |\nabla u|^2 dx = r^2, \quad f(x, u),\ g(x, u) \in L^1_{\mathrm{loc}}(\Omega),$$

under the assumptions that $sf(x, s) > 0$, $sg(x, s) > 0$ for $s \ne 0$ and

$$\sup_{|s|\le t} f(x, s), \quad \sup_{|s|\le t} g(x, s) \in L^1_{\mathrm{loc}}(\Omega)$$

for every $t > 0$ and $g(x, s)$ has at most a polynomial growth as $s \to \infty$.

In the present paper, we do not assume the condition that $sf(x, s) > 0$ or $sg(x, s) > 0$ for $s \ne 0$ (see Example 1.2(ii), (iii)) and do not need any growth condition on $g(x, s)$ as $s \to \infty$. We assume the Hölder continuity of f and g for the $C^2(\overline{\Omega})$ regularity of solutions. Even if we do not assume this condition, Theorem 1.1 is still valid after replacing the $C^2(\overline{\Omega})$ norm by the $C^1(\overline{\Omega})$ norm. We emphasize that the nonlinear term $f(x, u)$ in our paper is more general than those of the papers above and our theorem does not need any growth or sign condition on $g(x, u)$. To prove Theorem 1.1, we develop a new variational method based on the symmetric mountain pass lemma under the next assumption.

Assumption (B). Let E be an infinite dimensional Banach space and $I \in C([0, 1]\times E, \mathbb{R})$. Suppose that $I(t, u)$ has a continuous partial derivative I_u and satisfies (B1)–(B5) below.

(B1) $\inf\{I(t, u) : t \in [0, 1],\ u \in E\} > -\infty$.
(B2) There exists a function $\psi \in C([0, 1], \mathbb{R})$ such that $\psi(0) = 0$ and

$$|I(t, u) - I(0, u)| \le \psi(t) \quad \text{for } (t, u) \in [0, 1] \times E.$$

(B3) $I(t, u)$ satisfies the Palais–Smale condition uniformly on t, i.e. if a sequence (t_k, u_k) in $[0, 1] \times E$ satisfies that $\sup_k |I(t_k, u_k)| < \infty$ and $I_u(t_k, u_k)$ converges to zero, then (t_k, u_k) has a convergent subsequence.
(B4) $I(0, u) = I(0, -u)$ for $u \in E$ and $I(0, 0) = 0$.
(B5) For any $u \in E \setminus \{0\}$ there exists a unique $s(u) > 0$ such that $I(0, tu) < 0$ if $0 < |t| < s(u)$ and $I(0, tu) \ge 0$ if $|t| \ge s(u)$.

We define a critical value a_k of $I(0, u)$ in the following definition.

Definition 1.3. We set

$$S^k := \{x \in \mathbb{R}^{k+1} : |x| = 1\},$$
$$\mathcal{A}_k := \{h \in C(S^k, E) : h \text{ is odd}\},$$
$$a_k := \inf_{h\in\mathcal{A}_k} \max_{x\in S^k} I(0, h(x)). \tag{1.2}$$

In Section 2, we shall prove that $a_k \le a_{k+1} < 0$ for $k \in \mathbb{N}$ and $\{a_k\}$ converges to zero. Hence there exist infinitely many k's satisfying $a_k < a_{k+1}$, and so the next theorem makes sense.