J. Math. Anal. Appl. 400 (2013) 133–142

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Families of Gauss indicatrices on Lorentzian hypersurfaces in pseudo-spheres in semi-Euclidean 4-space

Jianguo Sun [a,b], Donghe Pei [a,*]

[a] School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, PR China

[b] School of Mathematics and Computational Science, China University of Petroleum (East China), Qingdao 266555, PR China

ARTICLE INFO

Article history:
Received 23 November 2011
Available online 23 October 2012
Submitted by R. Gornet

Keywords:
Pseudo-spheres
Gauss indicatrices
Lorentzian hypersurface
Height function
Singularity

ABSTRACT

We consider the one-parameter families of Gauss indicatrices on Lorentzian hypersurfaces in pseudo-spheres in semi-Euclidean 4-space with index 2 and give the types of singularities of the Lorentzian hypersurfaces by the contact theory.

1. Introduction

Since Einstein presented his theory of relativity, many scientists have been interested in studying the extrinsic differential geometry of submanifolds in semi-Euclidean space [1–11]. The difference between Euclidean space and semi-Euclidean space is the appearance of a light cone. The Gauss map is only spacelike in Euclidean space, but there exist a spacelike Gauss map and a hyperbolic Gauss map in semi-Euclidean space. The hyperbolic Gauss map can draw forth many new properties of geometry in semi-Euclidean space [4]. Against this background, the Minkowski space and semi-Euclidean space of index two are mainly considered by scientists. The properties of the differential geometry of many submanifolds in Minkowski space have been widely studied [1,3–7,10,11]. It contains three pseudo-spheres in semi-Euclidean space: the de Sitter sphere, the anti-de-Sitter sphere and the light cone. Lorentzian hypersurfaces and lightlike hypersurfaces in pseudo-spheres in semi-Euclidean space of index two have been also studied [2,8,9].

Legendrian dualities for pseudo-spheres in semi-Euclidean space give a commutative diagram between contact manifolds defined by the dual relations. From commutative diagrams [1–3,6,7], the differential geometry of spacelike hypersurfaces in one pseudo-sphere can be studied via the dual submanifolds in the other pseudo-spheres. Izumiya et al. defined two Gauss indicatrices: the de Sitter Gauss indicatrix and the hyperbolic Gauss indicatrix [4]. The flat geometry of the one-parameter form between the two Gauss indicatrices was called slant geometry [1,7].

The present study was inspired by Tari, who considered families of Gauss indicatrices on the hypersurfaces of a hyperbolic sphere and the timelike hypersurfaces of a de Sitter sphere in Minkowski 4-space [10]. Here we consider Lorentzian hypersurfaces on pseudo-spheres in semi-Euclidean space. For an index of two we have two cases: a de Sitter sphere $\mathbb{S}^3_2$ and an anti-de-Sitter sphere $\mathbb{H}^3_1$. We unify the two cases using Legendrian dualities. Therefore, we mainly consider families of Gauss indicatrices on Lorentzian hypersurfaces on an anti-de-Sitter sphere $\mathbb{H}^3_1$ and then provide a relation between two Gauss indicatrices, spacelike Gauss indicatrices N^s_θ and timelike Gauss indicatrices N^t_θ.

* Corresponding author.
E-mail addresses: sunjg616@yahoo.cn (J. Sun), peidh340@nenu.edu.cn (D. Pei).

0022-247X/$ – see front matter Crown Copyright © 2012 Published by Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2012.10.043

In Section 2 we review the basic notions of semi-Euclidean space and Legendrian dualities [2]. In Section 3 we consider notions of Lorentzian hypersurfaces on an anti-de-Sitter sphere $\mathbb{H}^3_1$. We define the families of spacelike and timelike Gauss indicatrices, which lead to definitions of a θ^ω-parabolic set and a θ^ω-umbilic surface. We also introduce spacelike and timelike height functions on Lorentzian hypersurfaces. We show that θ^ω-parabolic sets are given by two equations (Theorem 3.3). We study the singularities of the foliations $k_i = \text{constant}$ $(i = 1, 2)$, which are picked up by the families of height functions and (3.2) (Theorems 3.4 and 3.5). We then demonstrate the singularities of θ^ω-asymptotic curves for a generic surfaces in $\mathbb{H}^3_1$ (Theorem 3.7). In Section 4 we consider Lorentzian hypersurfaces in $\mathbb{S}^3_2$. According to the Legendrian dualities, we have the same differential geometry properties and singularities as in Section 3.

We assume throughout the paper that all manifolds and maps are C^∞ unless explicitly stated otherwise.

2. Preliminaries

Let $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) | x_i \in \mathbb{R}\ (i = 1, 2, 3, 4)\}$ be a four-dimensional vector space. For any vectors $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ and $\boldsymbol{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{R}^4$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = -x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4$. $(\mathbb{R}^4, \langle, \rangle)$ is called a *four-dimensional semi-Euclidean space of index two*, denoted by $\mathbb{R}^4_2$.

A vector $\boldsymbol{x}$ in $\mathbb{R}^4_2 \setminus \{\boldsymbol{0}\}$ is called a *spacelike vector*, a *lightlike vector* or a *timelike vector* if $\langle \boldsymbol{x}, \boldsymbol{x}\rangle$ is positive, zero or negative, respectively. The *norm* of a vector $\boldsymbol{x} \in \mathbb{R}^4_2$ is defined as $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x}\rangle|}$. For any two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^4_2$, we say that $\boldsymbol{x}$ is *pseudo-perpendicular* to $\boldsymbol{y}$ if $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0$. For any vectors $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$, $\boldsymbol{y} = (y_1, y_2, y_3, y_4)$ and $\boldsymbol{z} = (z_1, z_2, z_3, z_4)$ in $\mathbb{R}^4_2$, we define the pseudo-vector product as

$$\boldsymbol{x} \wedge \boldsymbol{y} \wedge \boldsymbol{z} = \begin{vmatrix} -\mathbf{e}_1 & -\mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix},$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ is the canonical form of $\mathbb{R}^4_2$. We can easily show that $\langle \boldsymbol{a}, \boldsymbol{x} \wedge \boldsymbol{y} \wedge \boldsymbol{z}\rangle = \det(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$. For a real number c, we define the hyperplane with pseudo-normal vector $\boldsymbol{n}$ by $HP(\boldsymbol{n}, c) = \{\boldsymbol{x} \in \mathbb{R}^4_2 \mid \langle \boldsymbol{x}, \boldsymbol{n}\rangle = c\}$. We call $HP(\boldsymbol{n}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\boldsymbol{n}$ is timelike, spacelike or lightlike, respectively. In $\mathbb{R}^4_2$, we have

de Sitter sphere $\mathbb{S}^3_2 = \{\boldsymbol{x} \in \mathbb{R}^4_2 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 1\}$,
anti-de-Sitter sphere $\mathbb{H}^3_1 = \{\boldsymbol{x} \in \mathbb{R}^4_2 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = -1\}$,
open lightcone $\wedge^3_1 = \{\boldsymbol{x} \in \mathbb{R}^4_2 \setminus \{\boldsymbol{0}\} \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 0\}$,
$S^1_t \times S^2_s$ lightcone $S^1_t \times S^2_s = \{\boldsymbol{x} \in \wedge^3_1 \mid x_0^2 + x_1^2 = x_2^2 + x_3^2 = 1\}$,

where $\boldsymbol{x} = (x_0, x_1, \dots, x_n)$, S^1_t denotes a timelike circle and S^2_s denotes a spacelike 2-sphere. Given any lightlike vector $\boldsymbol{x} = (x_0, x_1, x_2, x_3) \in \wedge^3_1$, we have

$$\widetilde{\mathbf{x}} = (x_0/\sqrt{x_0^2 + x_1^2}, x_1/\sqrt{x_0^2 + x_1^2}, \dots, x_3/\sqrt{x_0^2 + x_1^2}) \in S^1_t \times S^2_s.$$

We also consider contact manifolds and Legendrian submanifolds [6]. Let N be a $(2n + 1)$-dimensional smooth manifold and let K be a tangent hyperplane field on N. Locally, such a field is defined as the field of zeros of a 1-form α. The tangent hyperplane field K is non-degenerate if $\alpha \wedge (d\alpha)^n \neq 0$ at any point of N. We say that (N, K) is a *contact manifold* if K is a non-degenerate hyperplane field. In this case, K is called a *contact structure* and α is a contact form. Let $\phi : N \to N'$ be a diffeomorphism between contact manifolds (N, K) and (N', K'). We say that ϕ is a *contact diffeomorphism* if $d\phi(K) = K'$. Two contact manifolds (N, K) and (N', K') are *contact diffeomorphic* if there exists a contact diffeomorphism $\phi : N \to N'$. A submanifold $i : L \subset U$ of a contact manifold (N, K) is said to be *Legendrian* if $\dim L = n$ and $di_x(T_xL) \subset K_{i(x)}$ at any $x \in L$. We say that a smooth fiber bundle $\pi : E \to M$ is called a *Legendrian submanifold* if its total space E is furnished with a contact structure and its fibers are Legendrian submanifolds. Let $\pi : E \to M$ be a Legendrian fibration; for a Legendrian submanifold $i : L \subset E$, $\pi \circ i : L \to M$ is called a *Legendrian map*. The image of the Legendrian map $\pi \circ i$ is called a *wavefront set* of i, denoted by $W(L)$. For any $z \in E$, it is known that there is a local coordinate system $(x, y, p) = (x_1, \dots, x_m, y, p_1, \dots, p_m)$ around z such that $\pi(x, y, p) = (x, y)$ and the contact structure is given by the 1-form $\alpha = dy - \Sigma_{i=1}^m p_i dx_i$ [6]. We previously showed that the basic duality theorem is a fundamental tool for the study of Lorentzian hypersurfaces in pseudo-spheres in semi-Euclidean space [2]. Izumiya and Yıldırım considered ten contact manifolds [6]. However, we only need the Legendrian dualities (Δ_1, K_1), $(\Delta_2(\theta), K_2(\theta))$, $(\Delta_3(\theta), K_3(\theta))$ as follows:

(1) (a) $\mathbb{H}^n_1(-1) \times \mathbb{S}^n_2 \supset \Delta_1 = \{(\boldsymbol{v}, \boldsymbol{\omega}) \mid \langle \boldsymbol{v}, \boldsymbol{\omega}\rangle = 0\}$,
(b) $\pi_{11} : \Delta_1 \to \mathbb{H}^n_1(-1)$, $\pi_{12} : \Delta_1 \to \mathbb{S}^n_2$,
(c) $\eta_{11} = \langle d\boldsymbol{v}, \boldsymbol{\omega}\rangle |_{\Delta_1}$, $\eta_{12} = \langle \boldsymbol{v}, d\boldsymbol{\omega}\rangle |_{\Delta_1}$.

(2) (a) $\mathbb{H}^n_1(-1) \times \mathbb{H}^n_1(-(\sinh\theta)^{-2}) \supset \Delta_2(\theta) = \{(\boldsymbol{v}, \boldsymbol{\omega}) \mid \langle \boldsymbol{v}, \boldsymbol{\omega}\rangle = -\tanh^{-1}\theta\}$,
(b) $\pi_{21}(\theta) : \Delta_2(\theta) \to \mathbb{H}^n_1(-1)$, $\pi_{22}(\theta) : \Delta_2(\theta) \to \mathbb{H}^n_1(-(\sinh\theta)^{-2})$,
(c) $\eta_{21}(\theta) = \langle d\boldsymbol{v}, \boldsymbol{\omega}\rangle |_{\Delta_2(\theta)}$, $\eta_{22}(\theta) = \langle \boldsymbol{v}, d\boldsymbol{\omega}\rangle |_{\Delta_2(\theta)}$.