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# Dynamics for a complex-valued heat equation with an inverse nonlinearity

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ABSTRACT

We study the Cauchy problem for a parabolic system which is derived from a complex-valued heat equation with an inverse nonlinearity. First, we provide some criteria for the global existence of solutions. Then we consider the case when the initial data are asymptotically constants and obtain that, depending on the asymptotic limits, the solution quenches at space infinity or exists globally in time.



## 1. Introduction

In this paper, we study the following equation

$$z_t = z_{xx} - \frac{1}{z}, \tag{1.1}$$

where $z = z(x,t)$ is a complex-valued function of the spatial variable $x \in \mathbb{R}$ and the time variable $t \geq 0$. If we set $z(x,t) = u(x,t) + iv(x,t)$, where $i = \sqrt{-1}$ and $u(x,t), v(x,t) \in \mathbb{R}$, then (1.1) can be written as a system of parabolic equations

$$\begin{cases} u_t = u_{xx} - u/(u^2+v^2), \\ v_t = v_{xx} + v/(u^2+v^2). \end{cases} \tag{1.2}$$

If $z(x,t)$ is real-valued (i.e., $v \equiv 0$), then the system is reduced to the equation

$$u_t = u_{xx} - \frac{1}{u}.$$

An initial boundary value problem for the above equation was first studied by Kawarada [7] in 1975. For more general negative power nonlinearity, we refer the reader to, e.g., [4,6,8] and the references cited therein. The goal of this paper is to study the dynamics of solutions of the system (1.2) with $v \not\equiv 0$.

First of all, we consider a spatially homogeneous solution of (1.2), namely, $(u,v) = (U(t), V(t))$. We obtain that $(U(t), V(t))$ satisfies the following ODE system:

$$\begin{cases} U_t = -U/(U^2+V^2), \\ V_t = V/(U^2+V^2). \end{cases} \tag{1.3}$$

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Given $(U(0), V(0)) \in \mathbb{R}^2 \setminus \{(0,0)\}$. By a simple computation, we obtain that

$$U(t)V(t) = U(0)V(0) := C, \quad \forall t \geq 0, \tag{1.4}$$

for some constant $C \in \mathbb{R}$.

If $U(0) = 0$, then the trajectory stays on the $V$-axis, exists globally and tends to $\pm\infty$ as $t \to \infty$. On the other hand, if $V(0) = 0$, then $V(t) \equiv 0$ and $U$ tends to zero in finite time. When $C \neq 0$, by (1.3) and (1.4) we have $(U(t), V(t)) \to (0, \pm\infty)$ as $t \to \infty$.

In this paper, we consider the initial value problem (P) for (1.2) with the initial condition

$$(u(\cdot, 0), v(\cdot, 0)) = (u_0, v_0). \tag{1.5}$$

In the sequel, we shall always assume that

$$u_0 > 0, v_0 \geq 0, u_0, v_0 \in L^\infty(\mathbb{R}) \cap C(\mathbb{R}), \quad \inf_{\mathbb{R}} u_0 + \inf_{\mathbb{R}} v_0 > 0.$$

Then the problem (P) has a unique solution $(u, v) \in (C([0, T); L^\infty(\mathbb{R})))^2$, where $T = T(u_0, v_0) \in (0, \infty]$ is the maximal existence time of the solution. Furthermore, we have either $T = \infty$, or

$$T < \infty \quad \text{and} \quad \liminf_{t \to T} \{\inf_{x \in \mathbb{R}} u(x,t) + \inf_{x \in \mathbb{R}} v(x,t)\} = 0.$$

In the first case, we have the *global* existence. For the second case, we say that the solution of (P) *quenches* in a finite time $T$ in which $T$ is called the *quenching time*. Moreover, we say that $x_Q \in \mathbb{R}$ is a (finite) *quenching point* for $(u, v)$ if there exists a sequence $\{(x_j, t_j)\}$ such that $x_j \to x_Q$, $t_j \uparrow T$ and $u(x_j, t_j) + v(x_j, t_j) \to 0$ as $j \to \infty$. We shall investigate the global and non-global existence of solutions of (P).

The first result is about the global existence and (time) asymptotic behavior of the solution of the problem (P).

**Theorem 1.** *Suppose that the initial data satisfy*

$$\begin{cases} u_0(x) > 0, \quad v_0(x) > 0, \quad \forall x \in \mathbb{R}, \; u_0 \text{ and } v_0 \text{ are bounded in } \mathbb{R}, \\ u_0(x)v_0(x) \geq K, \quad \forall x \in \mathbb{R}, \; \text{for some constant } K > 0. \end{cases} \tag{1.6}$$

*Then the solution of (1.2) with (1.5) exists globally in time and $(u, v)$ converges to $(0, \infty)$ as $t \to \infty$ uniformly in $\mathbb{R}$.*

For $t \geq 0$, we set

$$\mathcal{R}(t) := \left\{(u(x,t), v(x,t)) \in \mathbb{R}^2; \; x \in \mathbb{R}\right\}$$

to be the image of the solution on $(u, v)$-plane. We remark that, under the hypothesis of Theorem 1, the closure of the convex hull of $\mathcal{R}(0)$ lies in the first quadrant of $(u, v)$-plane. Indeed, under the condition (1.6), we shall see that $\mathcal{R}(t)$ stays in the first quadrant for all $t > 0$. This implies the global existence of solutions.

On the other hand, if the initial data do not satisfy (1.6), in view of the dynamics of (1.3), it is interesting to see what happens. One question is to see under what conditions the quenching occurs. From (1.2) it is easy to see that both $u$ and $v$ quench simultaneously whenever quenching occurs. On the contrary, there might be *non-simultaneous quenching* in which just one component quenches and the other remains bounded away from zero. For this, we refer the reader to, e.g., [1,9,11,12,15].

To find solutions quenching in finite time, we consider the case when the initial data are *asymptotically constants*. Namely, we impose the following conditions on initial data:

$$u_0, v_0 \in C^1(\mathbb{R}), \quad u_0 \geq M, \quad u_0 \not\equiv M, \quad v_0 \geq 0, \quad v_0 \not\equiv 0, \tag{1.7}$$

$$\lim_{|x| \to \infty} u_0(x) = M, \quad \lim_{|x| \to \infty} v_0(x) = N \tag{1.8}$$

for some constants $M > 0$ and $N \geq 0$.

The following theorem shows that the solution of (1.2) with initial data satisfying (1.7) and (1.8) with $N > 0$ behaves like the solution the ODE system (1.3) with $(U(0), V(0)) = (M, N)$.

**Theorem 2.** *Let $(u, v)$ be a solution of (1.2) with initial data $(u_0, v_0)$ satisfying (1.7) and (1.8). If $N > 0$, then the solution of (1.2) with (1.5) exists globally for all $t \geq 0$ and $(u, v)$ converges to $(0, \infty)$ as $t \to \infty$ uniformly in $\mathbb{R}$.*

On the other hand, if the initial data of (1.2) satisfy (1.7) and (1.8) with $N = 0$, then the solution of (1.2) and (1.5) quenches only at space infinity. Namely, there are no (finite) quenching points, while there exists a sequence $\{(x_j, t_j)\}$ such that $|x_j| \to \infty$, $t_j \uparrow T$ and $u(x_j, t_j) + v(x_j, t_j) \to 0$ as $j \to \infty$.

**Theorem 3.** *Let $(u, v)$ be a solution of (1.2) and (1.5) with the initial data $(u_0, v_0)$ satisfying (1.7) and (1.8) with $M > 0$ and $N = 0$. Then the solution of (1.2) with (1.5) quenches at the finite time $t = T = M^2/2$. Moreover, the solution quenches only at space infinity.*