J. Math. Anal. Appl. 400 (2013) 161–173

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Eigenfunctions of a weighted Laplace operator in the whole space

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ARTICLE INFO

Article history:
Received 20 February 2012
Available online 23 June 2012
Submitted by Mr. V. Radulescu

Keywords:
Laplace operator
Unbounded domains
Rational functions
Stereographic projection
Weighted spaces

ABSTRACT

We study the spectrum of the weighted Laplacian $\varrho^{-1}\Delta$ in the whole space $\mathbb{R}^n$. We prove, under adequate conditions on ϱ^{-1}, that this spectrum is discrete and we derive an explicit formula for eigenvalues and eigenfunctions when $\varrho^{-1} = (|x|^2+1)^2$. We get by the way a complete family of rational functions which are mutually orthogonal in a weighted L^2 space.

1. Introduction

In this paper, we consider the eigenvalue problem

$$-\Delta u = \lambda \varrho u \quad \text{in } \mathbb{R}^n, \tag{1}$$

where ϱ is a given function. Our objective is twofold. First, we prove that problem (1), under some decay assumptions on ϱ at large distances, has an infinite family of eigenfunctions satisfying

$$\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^2+1} < +\infty, \qquad \int_{\mathbb{R}^n} |\nabla u|^2 dx < +\infty.$$

In a second part of the paper, we focus our attention on the particular case $\varrho(x) = (|x|^2+1)^{-2}$ for which explicit formula of eigenvalues and of eigenfunctions can be deduced. These eigenfunctions turn out to be rational functions.

We use throughout the paper weighted Sobolev spaces as a functional framework for describing the asymptotic behavior of functions.

We need to introduce some notations. Given an integer $n \geq 1$ and a typical point $x = (x_1, \ldots, x_n)$ of $\mathbb{R}^n$, we shall write

$$|x| = (x_1^2 + \cdots + x_n^2)^{1/2}, \qquad \langle x\rangle = (|x|^2+1)^{1/2}, \qquad \langle\langle x\rangle\rangle = \log(|x|^2+2).$$

By $L^p(\mathbb{R}^n)$, $p > 1$, we mean the usual Lebesgue space of pth-power integrable (class of) functions on $\mathbb{R}^n$, equipped with its usual norm. The symbol $\langle\cdot,\cdot\rangle$ will be used to designate duality pairing.

Let us define the weighted spaces we use here: given an integer $m \geq 0$ and a real α we set

$$W_\alpha^{m,p}(\mathbb{R}^n) = \{u \in \mathscr{D}'(\mathbb{R}^n);\ \forall\lambda \in \mathbb{N}^n,\ 0 \leq |\lambda| \leq m,\ \langle x\rangle^{\alpha-m+|\lambda|}\partial^\lambda u \in L^p(\mathbb{R}^n)\}.$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.06.009

This is a Banach space when equipped with the norm

$$\|u\|_{W^{m,p}_\alpha(\mathbb{R}^n)} = \left(\sum_{|\lambda|\le m} \|\langle x\rangle^{\alpha-m+|\lambda|}\partial^\lambda u\|_p^p \right)^{1/p}.$$

When $p = 2$, $W^{m,2}_\alpha(\mathbb{R}^n)$ is denoted by $W^m_\alpha(\mathbb{R}^n)$ (p is dropped). The reader can refer to, e.g., Hanouzet [1], Giroire [2–6], and references therein for more details. The following properties hold.

- The space $\mathscr{D}(\mathbb{R}^n)$ is dense in $W^{m,p}_\alpha(\mathbb{R}^n)$.
- The mapping $u \in W^{m,p}_\alpha(\mathbb{R}^n) \to \langle x\rangle^\beta u \in W^{m,p}_{\alpha-\beta}(\mathbb{R}^n)$, β is being a real, is an isomorphism.
- For any multi-index λ with $|\lambda| \le m$, the mapping $u \in W^{m,p}_\alpha(\mathbb{R}^n) \to \partial^\lambda u \in W^{m-|\lambda|,p}_\alpha(\mathbb{R}^n)$, is continuous.
- The following embeddings hold $W^{m,p}_\alpha(\mathbb{R}^n) \hookrightarrow W^{m-1,p}_{\alpha-1}(\mathbb{R}^n) \hookrightarrow \cdots \hookrightarrow W^{0,p}_{\alpha-m}(\mathbb{R}^n)$.
- Denoting by $\mathbb{P}_\ell$ the space of polynomials of degree less or equal to ℓ, $\ell \in \mathbb{Z}$ (with the convention $\mathbb{P}_\ell = \{0\}$ when $\ell < 0$), one has

$$\mathbb{P}_\ell \subset W^{m,p}_k(\mathbb{R}^n) \quad \text{when } \ell < m - k - \frac{n}{p}.$$

In the bidimensional case ($n = 2$), we need to introduce the space $X^1_0(\mathbb{R}^2)$ composed of all the distributions u satisfying

$$\int_{\mathbb{R}^2} \frac{|u|^2}{\langle x\rangle^2 \langle\langle x\rangle\rangle^2} dx < +\infty, \qquad \int_{\mathbb{R}^2} |\nabla u|^2 dx < +\infty.$$

This space is equipped with its natural norm. The dual of $X^1_0(\mathbb{R}^2)$ is denoted by $X^{-1}_0(\mathbb{R}^2)$. Observe that $X^1_0(\mathbb{R}^2) \not\subset W^0_{-1}(\mathbb{R}^2)$. However,

$$X^1_0(\mathbb{R}^2) \hookrightarrow W^0_{-\alpha}(\mathbb{R}^2),$$

for $\alpha > 1$. Thus, $W^0_\alpha(\mathbb{R}^2) \hookrightarrow X^{-1}_0(\mathbb{R}^2)$ for $\alpha > 1$.

In order to characterize eigenfunctions of the operator $\varrho^{-1}\Delta$ in the particular case $\varrho(x) = (1 + |x|^2)^{-2}$, we need for later use some notations concerning spherical harmonics on the unit sphere of $\mathbb{R}^{n+1}$, $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1};\ |x| = 1\}$. Recall that spherical harmonics of degree k are restrictions to the sphere $\mathbb{S}^n$ of harmonic homogeneous polynomials of degree k (see, e.g., [7] or [8]). Let $\mathbb{H}_k$ the space of spherical harmonics of degree k on $\mathbb{S}^n$, $k \ge 0$. We know that $L^2(\mathbb{S}^n) = \oplus_{k\ge 0} \mathbb{H}_k$, where $L^2(\mathbb{S}^n)$ is the usual space of (a class of) functions which are square integrable on the unit sphere. This space is equipped with the inner product

$$(u, v)_{L^2(\mathbb{S}^n)} = \int_{\mathbb{S}^n} u(\xi)\overline{v(\xi)} d\xi. \tag{2}$$

We set $d_k = \dim \mathbb{H}_k$. When $n = 1$, $d_k = 1$ for $k \ge 0$. When $n \ge 2$, we know that $d_0 = 1$, $d_1 = n + 1$ and

$$d_k = \binom{n+k}{n} - \binom{n+k-2}{n} = \binom{n+k-1}{k} + \binom{n+k-2}{k-1}, \quad \text{for } k \ge 2. \tag{3}$$

For each $k \ge 0$, we denote by $(\mathscr{Y}_{k,m})_{1\le m\le d_k}$ an orthogonal basis of $\mathbb{H}_k$ with respect to the inner product (2). We know that

$$-\Delta_{\mathbb{S}^n} \mathscr{Y}_{k,m} = k(k+n-1)\mathscr{Y}_{k,m} \quad \text{for } k \ge 0 \text{ and } 1 \le m \le d_k,$$

where $\Delta_{\mathbb{S}^n}$ is the usual Laplace–Beltrami operator.

The remaining of the paper is organized as follows. Section 2 is devoted to the first main result, that is the existence of a discrete spectrum of the operator $\varrho^{-1}\Delta$, under some conditions on ϱ. In Section 3, expressions of eigenvalues and eigenfunctions of the operator $(|x|^2 + 1)^2\Delta$ are derived by means of the stereographic projection. In Section 4, we disclose some important properties of the obtained eigenfunctions.

2. The first main result

In the sequel, ϱ denotes a non zero real measurable function. For any real s, we set

$$c^\star(\varrho; s) = \operatorname*{ess\,sup}_{\mathbb{R}^n} (|x|^2 + 1)^s \varrho(x), \qquad c_\star(\varrho; s) = \operatorname*{ess\,inf}_{\mathbb{R}^n} (|x|^2 + 1)^s \varrho(x).$$

We suppose that there exists a real $r > 1$ such that

$$0 < c_\star(\varrho; r) \le c^\star(\varrho; r) < +\infty. \tag{4}$$