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Analysis of a causal diffusion model and its backwards diffusion problem

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ABSTRACT

In Kowar (2012) [25] a diffusion model was developed and analyzed that obeys causality, i.e. the speed of propagation of the concentration is finite. In this article we analyze the respective causal backwards diffusion problem. The motivation for this paper is that because real diffusion obeys causality, a causal diffusion model may contain smaller modeling errors than the noncausal standard model and thus an increase of resolution of inverse and ill-posed problems related to diffusion is possible. We derive an analytic representation of the Green function of causal diffusion in the $\mathbf{k} - t$-domain (wave vector-time domain) that enables us to analyze the properties of the causal backwards diffusion problem. In particular, it is proven that this inverse problem is ill-posed, but not exponentially ill-posed. Furthermore, a theoretical and numerical comparison between the standard diffusion model and the causal diffusion model is performed. The paper is concluded with numerical simulations of the backwards diffusion problem via the Landweber method that confirm our theoretical results.

1. Introduction

Inverse problems related to diffusion belong to a very important field of applications. Such and similar problems are studied in the articles [1–14] and books [15–24] to name but a few. Since these inverse problems are *ill-posed*, *data* and *qualitative modeling errors* have a strong impact on the numerical solution. Hence if the diffusion model, the direct problem, is qualitatively improved, then it is possible to increase the resolution of related inverse problems. Since real diffusion has a finite diffusion speed, it is possible that the causal diffusion model developed in [25] has smaller modeling errors than the noncausal standard diffusion model. (Of course this can be verified by experiment only.) In this paper we focus on the analysis of the backwards diffusion problem corresponding to the *causal* diffusion model (direct problem) presented in [25]. In order to describe the goal of this paper, we start with a short description of the direct problem.

1.1. The direct problem

Let v denote the concentration of a substance diffusing with constant speed c and *initial concentration* u. We show that the causal diffusion model developed in [25] has the following analytic representation[1]

$$\hat{v}(\mathbf{k}, m+s) = (2\,\pi)^{-N/2}\Upsilon_N(|\mathbf{k}|)^m\,\Upsilon_N(|\mathbf{k}|\,s)\hat{u}(\mathbf{k}) \tag{1}$$

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[1] We will see that causal diffusion is determined by the speed of diffusion c, a time period τ and the space dimension N. Here we assume $c = 1$ and $\tau = 1$.

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for $\mathbf{k} \in \mathbb{R}^N$ $(N \in \mathbb{N})$, $m \in \mathbb{N}_0$ and $s \in (0, 1]$, where $\hat{v}$ denotes the Fourier transform of v with respect to $\mathbf{x} \in \mathbb{R}^N$ and Υ_N is the solution of

$$\Upsilon_N''(t) + \frac{(N-1)}{t}\,\Upsilon_N'(t) + \Upsilon_N(t) = 0 \quad t > 0, \tag{2}$$

with initial conditions

$$\Upsilon_N(0+) = 1 \quad \text{and} \quad \Upsilon_N'(0+) = 0. \tag{3}$$

For example, for $N = 1, 2, 3$ we have

$$\Upsilon_1(t) = \cos(t), \qquad \Upsilon_2(t) = \mathrm{J}_0(t) \quad \text{and} \quad \Upsilon_3(t) = \mathrm{sinc}(t), \tag{4}$$

where J_0 denotes the *Bessel function* of first kind and order zero (cf. [26]). Here $\check{\Upsilon}(\cdot, s)$ denotes the inverse Fourier transform of $\Upsilon(|\cdot|\, s)$.

1.2. The inverse problem

The respective backwards diffusion problem corresponds to the estimation of the *initial concentration* $u = u(\mathbf{x})$ for given concentration $w = w(\mathbf{x}) = v(\mathbf{x}, T)$ at time T. Mathematically this corresponds to the solution of the Fredholm integral equation of the first kind

$$F_T(u) = w \quad \text{for given data } w,$$

where the forward operator is defined by $F_T(u) := v(\cdot, T)$ with v as in (1) and $T > 0$ denotes the *data acquisition time*. We show (for appropriate spaces) that the forward operator is injective and that it is compact

(1) if $N = 2$ and $T > 2\,\tau$ and
(2) if $N \geq 3$ and $T > \tau$,

where N denotes the *space dimension* and t the time. In contrast to the standard diffusion model, the forward operator F_T is not compact for $N = 1$. Moreover, we show that the envelope of the Fourier transform of $F_T(u)$ does not decrease exponentially fast. In this sense the inverse problem is *not* exponentially ill-posed.

1.3. Irreversibility and time reversal

Numerical simulations for $N = 2$ indicate that the backwards diffusion problem is practically solvable for data acquisition time $T < \tau$ but not for $T \geq 3\tau$. This observation can be explained as follows: In [25, cf. Lemma 4 and Proposition 2] it is shown that v solves a hyperbolic equation of second order on each time interval $(n\tau, (n+1)\tau)$ for $n \in \mathbb{N}_0$ with "initial conditions"

$$v(\cdot, n\,\tau) = v(\cdot, n\,\tau+) \quad \text{and} \quad \frac{\partial v}{\partial t}(\cdot, n\,\tau+) = 0 \quad \left(\neq \frac{\partial v}{\partial t}(\cdot, n\,\tau-)\right).$$

But this means that – in contrast to the standard diffusion model (with exact data) – the technique of *time reversal* is not applicable, because the following initial data for the time reversal $(t \in [0, T] \to T - t)$

$$\frac{\partial v}{\partial t}(\cdot, \tau-),\ \frac{\partial v}{\partial t}(\cdot, 2\,\tau-), \ldots, \frac{\partial v}{\partial t}(\cdot, m\,\tau-), \ldots$$

cannot be acquired. If the data acquisition time satisfies $T < \tau$, then the (single) missing second initial data causes ill-posedness of the backwards diffusion problem, but if $T \approx m\tau$ with large $m \in \mathbb{N}$ than the backwards diffusion problem is unsolvable (even for zero noise). We consider this form of irreversibility of our diffusion model more physically reasonable than that one of the standard diffusion model.

In particular, this shows that a time reversal method cannot be used to solve the causal backwards diffusion problem studied in this paper. Therefore the numerical solution of the causal backwards diffusion problem is solved with an iterative regularization method.

The paper is organized as follows. In Section 2 we present our causal model of diffusion and derive those properties of diffusion that are needed for this paper. For the convenience of the reader we put the technical part in the Appendix. Comparisons between the standard diffusion model and our causal diffusion model are performed in Section 3. The theoretical and numerical aspects of the backwards diffusion problem are investigated in Sections 4 and 5. Numerical simulations of the inverse problem via the Landweber method are presented at the end of Section 5. The paper is concluded with a section of conclusion.