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Multidimensional fixed point theorems in partially ordered complete metric spaces

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ABSTRACT

In this paper we propose a notion of *coincidence point* between mappings in any number of variables and we prove some existence and uniqueness fixed point theorems for nonlinear mappings verifying different kinds of contractive conditions and defined on partially ordered metric spaces. These theorems extend and clarify very recent results that can be found in [T. Gnana-Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (7)(2006) 1379–1393], [V. Berinde, M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Anal. 74 (2011) 4889–4897] and [M. Berzig, B. Samet, An extension of coupled fixed point's concept in higher dimension and applications, Comput. Math. Appl. 63 (8) (2012) 1319–1334].

1. Introduction

The notion of *coupled fixed point* was introduced by Guo and Lakshmikantham [1] in 1987. In a recent paper, Gnana-Bhaskar and Lakshmikantham [2] introduced the concept of *mixed monotone property* for contractive operators of the form $F : X \times X \to X$, where X is a partially ordered metric space, and then established some coupled fixed point theorems. They also illustrated these results by proving the existence and uniqueness of the solution for a periodic boundary value problem. Later, Lakshmikantham and Ćirić [3] proved coupled coincidence and coupled common fixed point results for nonlinear mappings satisfying certain contractive conditions in partially ordered complete metric spaces. After that many results appeared on coupled fixed point theory in different contexts (see e.g. [4,3,5–7]).

The concept of *tripled fixed point* has been introduced by Berinde and Borcut [8]. In their manuscript, some new tripled point theorems are obtained using mixed monotone mappings. Their results generalize and extend the Gnana-Bhaskar and Lakshmikantham's research for nonlinear mappings. Moreover, these results could be used to study the existence of solutions of periodic boundary value problem involving $y'' = f(t, y, y')$. In [9], Turinici proved some product fixed point theorems in ordered metric spaces involving Picard operators and normal matrices. Then, he obtained a tripled fixed point theorem in Berinde and Borcut's sense. In fact, he proposed a contractivity condition using the maximum in order to improve Berinde and Borcut's argument.

Very recently, Berzig and Samet [10] have extended and generalized the mentioned fixed point results to higher dimensions. However, they used permutations of variables and distinguished between the first and the last variables. Furthermore, it is not clear the odd-dimensional case. Some authors [11] only refer Berinde and Borcut's paper, where we cannot deduce a simple way to choose the variables, for instance, in dimension five.

In this paper, our main aim is to obtain some existence and uniqueness theorems that extend the mentioned previous results for nonlinear mappings of any number of arguments, not necessarily permuted or ordered, in the framework of

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partially ordered complete metric spaces, using a weaker contraction condition. In order to do that, we propose a notion of *coincidence point* between mappings in any number of variables. Different kinds of contractive conditions are studied and we use a distinct methodology to prove our results. When there are two, three or four variables, particular cases of these results are already known under some contractive conditions. Finally, examples to support our results are also given.

2. Preliminaries

Let n be a positive integer. Henceforth, X will denote a non-empty set and X^n will denote the product space $X^n = X \times X \times \overset{n}{\cdots} \times X$. We represent the identity mapping on X as I_X. Throughout this manuscript, m and p will denote non-negative integers and $i, j, s \in \{1, 2, \dots, n\}$. Unless otherwise stated, "for all m" will mean "for all $m \geq 0$" and "for all i" will mean "for all $i \in \{1, 2, \dots, n\}$".

A *metric on* X is a mapping $d : X \times X \to \mathbb{R}$ satisfying, for all $x, y, z \in X$:

$$\text{(i) } d(x, y) = 0 \quad \text{if, and only if,} \quad x = y; \qquad \text{(ii) } d(x, y) \leq d(z, x) + d(z, y).$$

From these properties we can easily deduce that $d(x, y) \geq 0$ and $d(y, x) = d(x, y)$ for all $x, y \in X$. The last requirement is called the *triangle inequality*. If d is a metric on X, we say that (X, d) is a *metric space*.

Definition 1 (*[12]*)**.** A triple $(X, d, \leq)$ is called an *ordered metric space* if (X, d) is a metric space and $(X, \leq)$ is a partially ordered set.

Definition 2 (*[2]*)**.** Let $g : X \to X$ be a mapping. If $(X, d, \leq)$ is an ordered metric space, then X is said to have the *sequential g-monotone property* if it verifies the following properties:

(i) If $\{x_m\}$ is a non-decreasing sequence and $\lim_{m\to\infty} x_m = x$, then $gx_m \leq gx$ for all m.
(ii) If $\{y_m\}$ is a non-increasing sequence and $\lim_{m\to\infty} y_m = y$, then $gy_m \geq gy$ for all m.

If g is the identity mapping, then X is said to have the *sequential monotone property*.

Some authors introduced the concept of *coincidence point* in different ways and with different names. Let $F : X^n \to X$ and $g : X \to X$ be two mappings. For brevity, $g(x)$ will be denoted by gx.

Definition 3. A point $(x_1, x_2, \dots, x_n) \in X^n$ is

- a *coupled fixed point* [2] if $n = 2$, $F(x_1, x_2) = x_1$ and $F(x_2, x_1) = x_2$.
- a *tripled fixed point* [8] if $n = 3$, $F(x_1, x_2, x_3) = x_1$, $F(x_2, x_1, x_2) = x_2$ and $F(x_3, x_2, x_1) = x_3$.
- a *quartet fixed point* [13] if $n = 4$, $F(x_1, x_2, x_3, x_4) = gx_1$, $F(x_2, x_3, x_4, x_1) = gx_2$, $F(x_3, x_4, x_1, x_2) = gx_3$ and $F(x_4, x_1, x_2, x_3) = gx_4$.

3. The mixed g-monotone property and Φ-coincidence points

The following definitions extend previous considerations from other authors [2,8,13]. Let $F : X^n \to X$ and $g : X \to X$ be two mappings.

Definition 4. We say that F and g are *commuting* if $gF(x_1, x_2, \dots, x_n) = F(gx_1, gx_2, \dots, gx_n)$ for all $x_1, \dots, x_n \in X$.

Henceforth, fix a partition $\{A, B\}$ of $\Lambda_n = \{1, 2, \dots, n\}$, that is, $A \cup B = \Lambda_n$ and $A \cap B = \varnothing$. We will denote

$$\Omega_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq A \text{ and } \sigma(B) \subseteq B\}, \quad \text{and} \quad \Omega'_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq B \text{ and } \sigma(B) \subseteq A\}.$$

If $(X, \leq)$ is a partially ordered space, $x, y \in X$ and $i \in \Lambda_n$, we will use the following notation

$$x \leq_i y \Leftrightarrow \begin{cases} x \leq y, & \text{if } i \in A, \\ x \geq y, & \text{if } i \in B. \end{cases}$$

Definition 5. Let $(X, \leq)$ be a partially ordered space. We say that F has the *mixed g-monotone property* if F is g-monotone non-decreasing in arguments of A and g-monotone non-increasing in arguments of B, i.e., for all $x_1, x_2, \dots, x_n, y, z \in X$ and all i,

$$gy \leq gz \Rightarrow F(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) \leq_i F(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_n).$$

Henceforth, let $\sigma_1, \sigma_2, \dots, \sigma_n, \tau : \Lambda_n \to \Lambda_n$ be $n + 1$ mappings from Λ_n into itself and let Φ be the $(n + 1)$-tuple $(\sigma_1, \sigma_2, \dots, \sigma_n, \tau)$.