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Generalized Browder's theorem for tensor product and elementary operators

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ABSTRACT

The transfer property for the generalized Browder's theorem both of the tensor product and of the left-right multiplication operator will be characterized in terms of the *B*-Weyl spectrum inclusion. In addition, the isolated points of these two classes of operators will be fully characterized.

1. Introduction

In the recent past the relationship between, on the one hand, Weyl and Browder's theorems and their generalizations and, on the other, tensor products and elementary operators has been intensively studied, see for example [1–8]. In particular, given two operators that satisfy Browder's theorem, it is proved in [6] that a necessary and sufficient condition for the tensor product operator to satisfy Browder's theorem is that the Weyl spectrum identity holds, see the latter cited article or Section 4.

The main objective of this work is to characterize when given two operators that satisfy the generalized Browder's theorem, the tensor product operator also satisfies the generalized Browder's theorem, using in particular the *B*-Weyl spectrum identity. Furthermore, since one inclusion always holds for operators satisfying the generalized Browder's theorem, it is enough to consider the *B*-Weyl spectrum inclusion, see Section 4. It is worth noticing that since Browder's and the generalized Browder's theorem are equivalent [9], the results of this work also provide a characterization for the transfer property of the Browder's theorem for the tensor product operator.

However, to prove the key characterization of Section 4, the set of isolated points of the tensor product operator need to be studied. In particular, after Section 2 where several basic definitions and facts will be recalled, the poles and the complement of the poles in the isolated points of the tensor product operator will be characterized in terms of the corresponding sets of the source operators. It is important to note that these results continue and deepen the characterization of the isolated points of the tensor product operator presented in [3, see Section 3].

Finally, since the same arguments can be applied to the left–right multiplication operator, similar characterizations will be proved for elementary operators.

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2. Preliminary definitions

From now on $\mathcal{X}$ and $\mathcal{Y}$ shall denote infinite dimensional complex Banach spaces and $B(\mathcal{X}, \mathcal{Y})$ the algebra of all bounded linear maps defined on $\mathcal{X}$ and with values in $\mathcal{Y}$. As usual, when $\mathcal{X} = \mathcal{Y}$, $B(\mathcal{X}, \mathcal{X}) = B(\mathcal{X})$. Given $A \in B(\mathcal{X})$, $N(A)$, $R(A)$, $\sigma(A)$ and $\sigma_a(A)$ will stand for the null space, the range, the spectrum and the approximate point spectrum of A respectively. In addition, $\mathcal{X}^*$ will denote the dual space of $\mathcal{X}$, and if $A \in \mathcal{X}$, then $A^* \in B(\mathcal{X}^*)$ will stand for the adjoint map of A.

Recall that $A \in B(\mathcal{X})$ is said to be a *Weyl operator*, if the dimensions both of $N(A)$ and of $\mathcal{X}/R(A)$ are finite and equal. Let $\sigma_w(A)$ be the *Weyl spectrum of* A, i.e., $\sigma_w(A) = \{\lambda \in \mathbb{C}: A - \lambda \text{ is not Weyl}\}$, where $A - \lambda$ stands for $A - \lambda I$, I the identity map of $\mathcal{X}$. Note, in addition, that the concept of Weyl operator has been generalized recently. An operator $A \in B(\mathcal{X})$ will be said to be *B-Weyl*, if there exists $n \in \mathbb{N}$ for which the range of $R(A^n)$ is closed and the induced operator $A_n \in B(R(A^n))$ is Weyl [10]. It is worth noticing that if for some $n \in \mathbb{N}$, $A_n \in B(R(A^n))$ is Weyl, then $A_m \in B(R(A^m))$ is Weyl for all $m \geq n$ [11]. Naturally, from this class of operators the B-Weyl spectrum of $A \in B(\mathcal{X})$ can be derived in the usual way; this spectrum will be denoted by $\sigma_{BW}(A)$.

On the other hand, a Banach space operator $A \in B(\mathcal{X})$ is said to be *Drazin invertible*, if there exists a necessarily unique $B \in B(\mathcal{X})$ and some $m \in \mathbb{N}$ such that

$$A^m = A^m BA, \qquad BAB = B, \qquad AB = BA.$$

If $DR(B(\mathcal{X})) = \{A \in B(\mathcal{X}): A \text{ is Drazin invertible}\}$, then the *Drazin spectrum* of $A \in B(\mathcal{X})$ is the set $\sigma_{DR}(A) = \{\lambda \in \mathbb{C}: A - \lambda \notin DR(B(\mathcal{X}))\}$ [12,13].

The *ascent* (respectively *the descent*) of $A \in B(\mathcal{X})$ is the smallest non-negative integer $a = asc(A)$ (respectively $d = dsc(A)$) such that $N(A^a) = N(A^{a+1})$ (respectively $R(A^d) = R(A^{d+1})$); if such an integer does not exist, then $asc(A) = \infty$ (respectively $dsc(A) = \infty$). Recall that $\lambda \in \sigma(A)$ is said to be *a pole* of A, if the ascent and the descent of $A - \lambda$ are finite (hence equal). The set of poles of $A \in B(\mathcal{X})$ will be denoted by $\Pi(A)$. Note that $\Pi(A) = \sigma(A) \setminus \sigma_{DR}(A)$ [14, Theorem 4]. In particular, if $A \in B(\mathcal{X})$ is quasi-nilpotent, then according to [14, Theorem 5], necessary and sufficient for A to be nilpotent is that $\Pi(A) = \{0\}$. In addition, the set of *poles of finite rank* of A is the set $\Pi_0(A) = \{\lambda \in \Pi(A): \alpha(A - \lambda) < \infty\}$, where $\alpha(A - \lambda) = \dim N(A - \lambda)$.

Recall that an operator $A \in B(\mathcal{X})$ is said to satisfy *Browder's theorem*, if $\sigma_w(A) = \sigma(A) \setminus \Pi_0(A)$, while A is said to satisfy the *generalized Browder's theorem*, if $\sigma_{BW}(A) = \sigma(A) \setminus \Pi(A) = \sigma_{DR}(A)$. According to [9, Theorem 2.1], the Browder's and generalized Browder's theorems are equivalent. Moreover, according to [15, Theorem 2.1(iv)], the generalized Browder's theorem is equivalent to the fact that $\text{acc}\, \sigma(A) \subseteq \sigma_{BW}(A)$. Here and elsewhere in this article, for $K \subseteq \mathbb{C}$, $\text{iso}\, K$ will stand for the set of isolated points of K and $\text{acc}\, K = K \setminus \text{iso}\, K$ for the set of limit points of K. The generalized Browder's theorem was studied in [8,9,15–17].

In what follows, given Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{X}\overline{\otimes}\mathcal{Y}$ will stand for the completion, endowed with a reasonable uniform cross-norm, of the algebraic tensor product $\mathcal{X} \otimes \mathcal{Y}$ of $\mathcal{X}$ and $\mathcal{Y}$. In addition, if $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$, then $A \otimes B \in B(\mathcal{X}\overline{\otimes}\mathcal{Y})$ will denote the tensor product operator defined by A and B.

On the other hand, $\tau_{AB} \in B(B(\mathcal{Y}, \mathcal{X}))$ will denote the multiplication operator defined by $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$, i.e., $\tau_{AB}(U) = AUB$, where $U \in B(\mathcal{Y}, \mathcal{X})$ and $\mathcal{X}$ and $\mathcal{Y}$ are two Banach spaces. Note that $\tau_{AB} = L_A R_B$, where $L_A \in B(B(\mathcal{Y}, \mathcal{X}))$ and $R_B \in B(B(\mathcal{Y}, \mathcal{X}))$ are the left and right multiplication operators defined by A and B respectively, i.e., $L_A(U) = AU$ and $R_B(U) = UB$, $U \in B(\mathcal{Y}, \mathcal{X})$.

3. The isolated points

In this section the isolated points both of the tensor product and of the left–right multiplication operator will be studied. To this end, some preparation is needed.

Remark 3.1. Let $\mathcal{X}$ be a Banach space, consider $A \in B(\mathcal{X})$ and set $I(A) = \text{iso}\, \sigma(A) \setminus \Pi(A)$.

(i) Necessary and sufficient for $\lambda \in \sigma(A)$ to belong to $I(A)$ is that there exist M and N, two closed and complemented subspaces of $\mathcal{X}$ invariant for A, such that if $A_1 = A|_M$ and $A_2 = A|_N$, then $A_1 - \lambda$ is quasi-nilpotent but not nilpotent and $A_2 - \lambda$ is invertible. Note that $\sigma(A) = I(A) = \{\lambda\}$ if and only if $N = 0$.

(ii) Let $\lambda \in \sigma(A)$. The complex number λ belongs to $\Pi(A)$ if and only if there are M' and N' two closed and complemented subspaces of $\mathcal{X}$ invariant for A, such that if $A' = A|_{M'}$ and $A'' = A|_{N'}$, then $A' - \lambda$ is nilpotent and $A'' - \lambda$ is invertible. As in statement (i), $\sigma(A) = \Pi(A) = \{\lambda\}$ is equivalent to the fact that $N' = 0$.
Statements (i)–(ii) are well known and they can be easily deduced from [13, Theorem 12] and [14, Theorem 5]. Now let $\mathcal{Y}$ be a Banach space and consider $B \in B(\mathcal{Y})$.

(iii) Since $\sigma(A \otimes B) = \sigma(A)\sigma(B) = \sigma(\tau_{AB})$ [18, Theorem 2.1] and [19, Corollary 3.4], according to [3, Theorem 6],

$$(\text{iso}\, \sigma(A \otimes B)) \setminus \{0\} = (\text{iso}\, (\tau_{AB})) \setminus \{0\} = (\text{iso}\, \sigma(A) \setminus \{0\})(\text{iso}\, \sigma(B) \setminus \{0\}).$$

(iv) Set

$$\mathbb{L} = (I(A) \setminus \{0\})(I(B) \setminus \{0\}) \cup (I(A) \setminus \{0\})(\Pi(B) \setminus \{0\}) \cup (\Pi(A) \setminus \{0\})(I(B) \setminus \{0\}).$$

Then clearly, $(\text{iso}\, \sigma(A \otimes B)) \setminus \{0\} = (\text{iso}\, (\tau_{AB})) \setminus \{0\} = \mathbb{L} \cup (\Pi(A) \setminus \{0\})(\Pi(B) \setminus \{0\})$.