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An inverse problem by eigenvalues of four spectra

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ABSTRACT

Certain parts of the Dirichlet–Dirichlet, Neumann–Dirichlet, Dirichlet–Neumann and Neumann–Neumann spectra are used to find the potential of the Sturm–Liouville equation on a finite interval. This problem possesses a unique solution. Conditions are found necessary and sufficient for four sequences to be the corresponding parts of the four spectra.

1. Introduction

We consider the Sturm–Liouville boundary value problems with Dirichlet and Neumann boundary conditions on a finite interval $[0, a]$. By the Dirichlet–Dirichlet problem we mean the one with the Dirichlet conditions at both ends of the interval (see problem (2.1), (2.2)), by Neumann–Dirichlet the problem with the Neumann condition at the left end and the Dirichlet condition at the right end (see problem (2.1), (2.3)) and so on.

It is well known that the spectra of the Neumann–Dirichlet (or the Dirichlet–Neumann) and the Dirichlet–Dirichlet boundary value problems generated by the same potential uniquely determine this potential in $L_1(0, a)$. Also it is known that the spectra of two boundary value problems with the same Robin boundary condition at one of the ends and different Robin conditions at the other end of the interval uniquely determine the potential and the constants in the boundary conditions. These results are due to Borg [1] (see also [2–4]). If the boundary conditions are given data then the problem of recovering the potential appears to be overdetermined (in the case of Robin conditions). One needs to know not all the eigenvalues of the two spectra. This was shown in [5] and is sometimes called the 'missing eigenvalue problem' (see [6]). Further development of this theory lies in the use of one spectrum together with the knowledge of a part of the potential [7,5,8,6,9].

Another direction of generalization of the above results is to use eigenvalues of more than two spectra to determine the potential. In [10] it was shown that $2/3$ part of the union of three spectra of boundary value problems with the same boundary condition at one of the ends uniquely determine the potential. In [11] a similar but more general sufficient condition of unique solvability was given for the case when the known eigenvalues were taken from n different spectra (see [12] for a topical review).

In the present paper we consider real potentials from $L_2(0, a)$ which enables us to use interpolation in the Paley–Wiener class using the results of [13,14]. We use eigenvalues of four boundary value problems, namely the Dirichlet–Dirichlet, the Neumann–Dirichlet, the Dirichlet–Neumann and the Neumann–Neumann problem to recover the potential.

In Section 2 we describe some well known facts about interlacing properties of eigenvalues of the Dirichlet–Dirichlet, the Neumann–Dirichlet, the Dirichlet–Neumann and the Neumann–Neumann problem and the eigenvalue asymptotics

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of these problems. We reformulate known results of [4] in the form of a theorem on solvability and uniqueness of solution of a functional equation. This theorem is used in Section 3 where we prove that certain parts of the spectra of the Dirichlet–Dirichlet, the Neumann–Dirichlet, the Dirichlet–Neumann and the Neumann–Neumann problem uniquely determine the potential. We also characterize the given data of such inverse problem, i.e. we give conditions necessary and sufficient for four sequences of real numbers to be the squares of eigenvalues of certain parts of the spectra of the mentioned problems and describe the procedure of recovering the potential. We use the method that was earlier used in [15] to solve the three spectral inverse problem and in [16,17] to solve the inverse problem on a star graph.

2. Direct problems

Let us consider four boundary value problems with a real potential $q \in L_2(0, a)$: the Dirichlet–Dirichlet problem

$$-y'' + q(x)y = \lambda^2 y, \quad x \in [0, a] \tag{2.1}$$

$$y(0) = y(a) = 0, \tag{2.2}$$

the spectrum of which we denote by $\{\nu_k\}_{-\infty,\ k\neq 0}^{\infty}$ $(\nu_{-k} = -\nu_k)$, the Neumann–Dirichlet problem

$$-y'' + q(x)y = \lambda^2 y, \quad x \in [0, a]$$

$$y'(0) = y(a) = 0, \tag{2.3}$$

with the spectrum denoted by $\{\mu_k\}_{-\infty,\ k\neq 0}^{\infty}$ $(\mu_{-k} = -\mu_k)$, the Dirichlet–Neumann problem

$$-y'' + q(x)y = \lambda^2 y, \quad x \in [0, a]$$

$$y(0) = y'(a) = 0, \tag{2.4}$$

with the spectrum which we denote by $\{\kappa_k\}_{-\infty,\ k\neq 0}^{\infty}$ $(\kappa_{-k} = -\kappa_k)$, and the Neumann–Neumann problem

$$-y'' + q(x)y = \lambda^2 y, \quad x \in [0, a]$$

$$y'(0) = y'(a) = 0, \tag{2.5}$$

with the spectrum which we denote by $\{\zeta_k\}_{-\infty,\ k\neq 0}^{\infty} \cup \{\zeta_{-0}, \zeta_{+0}\}$ $(\zeta_{-k} = -\zeta_k)$. This way of enumeration appears to be convenient.

Let us denote by $s_j(\lambda, x)$ the solution of the Sturm–Liouville equation (2.1) which satisfies the conditions $s_j(\lambda, 0) = s_j'(\lambda, 0) - 1 = 0$ and by $c_j(\lambda, x)$ the solution which satisfies the conditions $c_j(\lambda, 0) - 1 = c_j'(\lambda, 0) = 0$. According to [4]

$$\begin{aligned} s(\lambda, x) &= \frac{\sin \lambda x}{\lambda} + \int_0^x K(x, t)\frac{\sin \lambda t}{\lambda} dt \\ &= \frac{\sin \lambda x}{\lambda} - K(x, x)\frac{\cos \lambda x}{\lambda^2} + \int_0^x K_t(x, t)\frac{\cos \lambda t}{\lambda^2} dt, \end{aligned} \tag{2.6}$$

where

$$K(x, t) = \tilde{K}(x, t) - \tilde{K}(x, -t), \qquad K_t(x, t) = \frac{\partial K(x, t)}{\partial t}$$

and $\tilde{K}(x, t)$ is the solution of the integral equation

$$\tilde{K}(x, t) = \frac{1}{2}\int_0^{\frac{x+t}{2}} q(s)ds + \int_0^{\frac{x+t}{2}} d\alpha \int_0^{\frac{x-t}{2}} q(\alpha + \beta)\tilde{K}(\alpha + \beta, \alpha - \beta)d\beta.$$

The solution $\tilde{K}(x, t)$ possesses partial derivatives of the first order each belonging to $L_2(0, a)$ as a function of each of its variables. Moreover, $K(x, 0) = 0$ and

$$K(x, x) = \frac{1}{2}\int_0^x q(t)dt.$$

It is clear also that

$$s'(\lambda, x) = \cos \lambda a + K(x, x)\frac{\sin \lambda a}{\lambda} + \int_0^x K_x(x, t)\frac{\sin \lambda t}{\lambda}, \tag{2.7}$$

$$\begin{aligned} c(\lambda, x) &= \cos \lambda x + \int_0^x B(x, t) \cos \lambda t dt \\ &= \cos \lambda x + B(x, x)\frac{\sin \lambda x}{\lambda} + \int_0^x B_t(x, t)\frac{\cos \lambda t}{\lambda} dt, \end{aligned} \tag{2.8}$$