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π-Formulas with free parameters

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ABSTRACT

In terms of the hypergeometric method, we establish ten general π-formulas with free parameters which include several known results as special cases.

1. Introduction

For a complex number x and an integer n, define the shifted factorial by

$$(x)_n = \begin{cases} \prod_{k=0}^{n-1}(x+k), & \text{when } n > 0; \\ 1, & \text{when } n = 0; \\ \dfrac{(-1)^n}{\prod_{k=1}^{-n}(k-x)}, & \text{when } n < 0. \end{cases}$$

Recall that the function $\Gamma(x)$ can be given by Euler's integral:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad \text{with } \mathrm{Re}(x) > 0.$$

Then we have the following two relations:

$$\Gamma(x+n) = \Gamma(x)(x)_n, \qquad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)},$$

which will frequently be used without indication in this paper.

Following Bailey [1], define the hypergeometric series by

$${}_{r+1}F_s\left[\begin{matrix} a_0, & a_1, & \ldots, & a_r \\ & b_1, & \ldots, & b_s \end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k\cdots(a_r)_k}{k!(b_1)_k\cdots(b_s)_k} z^k.$$

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Then a simple ${}_2F_1$-series identity (cf. [2, Eq. (26)]) can be stated as

$$ {}_2F_1\left[\begin{matrix}1, & \frac{1}{2}\\ & \frac{3}{2}\end{matrix}\middle| x^2\right] = \frac{\arcsin(x)}{x\sqrt{1-x^2}} \quad \text{where } |x| < 1. $$

Two beautiful series for π (cf. [2, Eqs. (23) and (27)]) implied by it read as

$$ \frac{\pi}{2} = \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!!}, \tag{1} $$

$$ \frac{2\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty} \frac{(k!)^2}{(2k+1)!}, \tag{2} $$

where the double factorial has been offered by

$$ (2k+1)!! = \frac{(2k+1)!}{2^k k!}, \qquad (2k)!! = 2^k k!. $$

By means of WZ-method, Guillera [3, p. 221] derive lately the nice series for π^2:

$$ \frac{\pi^2}{4} = \sum_{k=0}^{\infty} \frac{(1)_k^3}{\left(\frac{3}{2}\right)_k^3} \frac{3k+2}{4^k}. \tag{3} $$

Recall the ${}_7F_6$-series identity due to Chu [4, Eq. (5.1e)] and Dougall's ${}_5F_4$-series identity (cf. [1, p. 27]):

$$ {}_7F_6\left[\begin{matrix} a-\frac{1}{2}, & \frac{2a+2}{3}, & 2b-1, & 2c-1, & 2+2a-2b-2c, & a+s, & -s\\ & \frac{2a-1}{3}, & 1+a-b, & 1+a-c, & b+c-\frac{1}{2}, & 2a+2s, & -2s\end{matrix}\middle| 1\right] $$

$$ = \frac{\left(\frac{1}{2}+a\right)_s (b)_s (c)_s \left(a-b-c+\frac{3}{2}\right)_s}{\left(\frac{1}{2}\right)_s (1+a-b)_s (1+a-c)_s \left(b+c-\frac{1}{2}\right)_s} \tag{4} $$

where s is a positive integer,

$$ {}_5F_4\left[\begin{matrix} a, & 1+\frac{a}{2}, & b, & c, & d\\ & \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d\end{matrix}\middle| 1\right] $$

$$ = \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)} \tag{5} $$

provided that $\mathrm{Re}(1+a-b-c-d) > 0$.

Recently, Chu [5,6] and Liu [7,8] have deduced many surprising π-formulas from some known hypergeometric series identities. Thereinto, Chu [5] showed that (5) implies the Ramanujan-type series for $1/\pi$ with three free parameters:

$$ \frac{2}{\pi} = \frac{\left(\frac{1}{2}\right)_{m-n-p}}{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_p} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{2}\right)_{k+m} \left(\frac{1}{2}\right)_{k+n} \left(\frac{1}{2}\right)_{k+p}}{k!(k+m-n)!(k+m-p)!} (4k+2m+1) \tag{6} $$

where $m, n, p \in \mathbb{Z}$ with $\min\{m-n, m-p, m-2n-2p\} \geq 0$ and the Ramanujan-type series for $1/\pi^2$ with four free parameters:

$$ \frac{2}{\pi^2} = \frac{\left(\frac{1}{2}\right)_{m-n-p} \left(\frac{1}{2}\right)_{m-n-q} \left(\frac{1}{2}\right)_{m-p-q}}{(m-n-p-q-1)! \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_p \left(\frac{1}{2}\right)_q} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{2}\right)_{k+m} \left(\frac{1}{2}\right)_{k+n} \left(\frac{1}{2}\right)_{k+p} \left(\frac{1}{2}\right)_{k+q}}{k!(k+m-n)!(k+m-p)!(k+m-q)!} (4k+2m+1) \tag{7} $$

where $m, n, p, q \in \mathbb{Z}$ with $\min\{m-n, m-p, m-q, m-n-p-q-1\} \geq 0$. Liu [8] showed that (5) implies the Ramanujan-type series for $1/\pi$ with four free parameters:

$$ \frac{\sqrt{3}}{3\pi} = \frac{\left(\frac{2}{3}\right)_{m-n-p} \left(\frac{1}{3}\right)_{m-n-q} \left(\frac{1}{2}\right)_{m-p-q}}{(m-n-p-q-1)! \left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_p \left(\frac{2}{3}\right)_q} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{k+m} \left(\frac{1}{2}\right)_{k+n} \left(\frac{1}{3}\right)_{k+p} \left(\frac{2}{3}\right)_{k+q}}{k!(k+m-n)! \left(\frac{7}{6}\right)_{k+m-p} \left(\frac{5}{6}\right)_{k+m-q}} (4k+2m+1) \tag{8} $$