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Stochastic portfolio optimization with default risk

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ABSTRACT

A stochastic portfolio optimization problem with default risk on an infinite time horizon is investigated. The default risk premium and the default intensity corresponding to the defaultable bond are assumed to rely on a stochastic factor formulated by a diffusion process. We study the optimal allocation and consumption policies to maximize the infinite horizon expected discounted non-log HARA utility of the consumption, and we use the dynamic programming principle to derive the Hamilton–Jacobi–Bellman (HJB) equation. Then we explore the HJB equation by employing a so-called sub–super solution approach. The optimal allocation and consumption policies are finally presented in a verification theorem, and also a numerical simulation is given at the end of the paper.

1. Introduction

The seminal works by Merton [1–3] proposed the strategy that maximizing the total expected discounted utility of the consumption for a market investment problem. Since then, default-free portfolio optimization models have been extensively investigated in the literature (see, e.g., [4–11] and references therein). In [4], Fleming and Pang discussed a classical Merton portfolio optimization problem, where the interest rate r was assumed to be an ergodic Markov diffusion process. An analogue consideration with the log HARA utility was in [10]. In [11], Pham treated an optimal investment model with stochastic volatilities. There, the instantaneous rate r and the volatility of the stock's return were assumed to rely on a stochastic factor formulated by a diffusion process. However, unlike the classical Merton model, a closed form solution cannot be obtained for the derived HJB equation. Instead, a numerical method had to be adopted, once the existence of the classical solution to the HJB equation is proved under some appropriate conditions.

Recently, defaultable securities such as corporate bonds have been increasingly attractive to the investors due to more and more yields. As a consequence, the portfolio optimization problem with defaultable securities has become a more interesting topic (see, e.g., [12–15] and references therein). Among the literature, Bielecki and Jang [12] studied an optimal allocation problem associated with a defaultable risky asset and there the goal was to maximize the expected HARA utility of the terminal wealth. In [14], Hou and Jin employed an intensity-based approach for the defaultable market and assumed that each investor receives a proportion of the market value of the debt prior to the default if a default occurs. Jang [15] suggested a dynamics for the price of a defaultable bond, and studied the expected discounted utility of the wealth when the default risk premium and intensity were assumed to be constants. Bo et al. [13] considered a portfolio optimal problem with default risk under the intensity-based reduced-form framework and the goal was to maximize the infinite horizon expected discounted log utility of the consumption, where the model differs from [15], since the default risk premium and the default intensity were assumed to rely on a stochastic factor described by a diffusion process.

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In this article, we investigate a portfolio optimization problem with default risk. An investor dynamically chooses a consumption rate and allocates the wealth into the securities: a perpetual defaultable bond, a money market account with the constant return and a default-free risky asset. Here the goal is to maximize the infinite horizon expected discounted utility of the consumption. We deal with the non-log HARA utility function case, and the log utility counterpart has been discussed in [13]. There, the post-default HJB equation admitted a constant solution and the pre-default HJB equation is a linear uniformly elliptic equation with variable coefficients. For the non-log utility case, we find that the HJB equation is nonlinear. Due to its nonlinearity, we adopt the so-called sub–super solution argument to study the equation. Finally, we get an explicit formula for the optimal control strategy. The readers may refer to Mariani et al. [16,17] for solving PDE problems arising in financial mathematics.

The outline of the paper is as follows. In the coming section, we describe the model. In Section 3, the optimal portfolio problem with default risk is explored under the non-log HARA utility. Section 4 is devoted to proving a verification theorem. Finally, in Section 5, we carry out a sensitivity analysis for the optimal control strategy and the value function, respectively. Section 6 concludes the paper.

2. The model formulation

In this section, we shall present a model with the specifications of a reduced-form framework for an intensity-based defaultable market and of the dynamics of the financial securities (defaultable bond, money market account and default-free risky asset).

2.1. The reduced-form framework

In the subsection, we give a reduced-form framework for an intensity-based defaultable market.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete real-world probability space and τ be a nontrivial random time on the space. For $t \geq 0$, let us define a default indicator process $(z_t)_{t\geq 0}$ by

$$z_t = \mathbf{1}_{\{\tau \leq t\}}. \tag{2.1}$$

Suppose that $(\omega_t, \breve{\omega}_t)_{t\geq 0}$ is a 2-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the augmented natural filtration of $(\omega_t, \breve{\omega}_t)_{t\geq 0}$. Let $\mathcal{D}_t = \sigma(z_u; 0 \leq u \leq t)$ and $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$ with $t \geq 0$.

Consider the conditional survival probability,

$$S_t = \mathbb{P}(\tau > t | \mathcal{F}_t), \qquad S_0 = 1. \tag{2.2}$$

Assume that for each $t > 0, S_t > 0$ a.s. and $\mathbb{E}[S_t] > 0$. This implies that there is always a chance that the firm defaults. Due to the supermartingale property of the process $(S, \mathbb{F})$, the Doob–Meyer Theorem tells us that there exists a unique compensator K to S such that $(S + K, \mathbb{F})$ becomes a martingale. According to Giesecke [18], we can define the trend by the Stieltjes integral

$$\Lambda_t = \int_0^t \frac{\mathrm{d}K_s}{S_{s-}}, \tag{2.3}$$

which is a nondecreasing $\mathbb{F}$-predictable process. The definition of the reduced-form framework for an intensity-based defaultable market (see, e.g., [19,20,13,18]) goes as follows.

Definition 2.1. It is called the intensity-based reduced-form model if the trend $\Lambda_t = \int_0^t \lambda_u \mathrm{d}u$ with a nonnegative $\mathbb{F}$-predictable intensity process $\lambda = (\lambda_t)_{t\geq 0}$ for each $t \geq 0$.

In light of Propositions 5.8 and 5.2 in [18], it follows that $S_t = \exp\left(-\int_0^t \lambda_s \mathrm{d}s\right)$ and

$$m_t := z_t - \int_{]0, t\wedge\tau]} \lambda_s \mathrm{d}s, \quad t \geq 0 \tag{2.4}$$

is a $(\mathbb{P}, \mathbb{G})$-martingale with the information $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$.

2.2. The price dynamics of the financial securities

Let $(1/\eta_t)_{t\geq 0}$ denote the default risk premium satisfying $1/\eta_t \geq 1$ for all $t \geq 0$, and $\rho \in (0, 1]$ denote the constant loss rate when a default occurs. We can suggest the price dynamics $(p_t)_{t\geq 0}$ for a perpetual defaultable bond that pays constant coupon $\widetilde{C}$ per unit time as follows[1]:

$$\mathrm{d}p_t = rp_t\mathrm{d}t + \rho\lambda_t p_t(1 - z_t)(1/\eta_t - 1)\mathrm{d}t - (1 - z_t)\widetilde{C}\mathrm{d}t - \rho p_{t-}\mathrm{d}m_t, \tag{2.5}$$

1 The derivation of the dynamics when the market parameters are constant is given in Appendix A of [13]. Here we directly randomize the market parameters in the dynamics as the manner used by the stochastic volatility model (see, e.g., [21]). One must be careful when the market parameters are random, since then the derivation in Appendix A of [13] will be invalid.