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On the value distribution of some difference polynomials[☆]

Xiu-Min Zheng[a,*], Zong-Xuan Chen[b]

[a] Institute of Mathematics and Information Science, Jiangxi Normal University, Nanchang, 330022, China
[b] School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, China

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ABSTRACT

In this paper, we continue to investigate the value distribution of some difference polynomials, which can be considered as difference counterparts of some classic results of Hayman.

1. Introduction and results

Throughout this paper, we use standard notations in the Nevanlinna theory (see e.g. [12,16,20]). Let $f(z)$ be a meromorphic function. Here and in the following the word "meromorphic" means meromorphic in the whole complex plane. We use notations $\sigma(f)$ and $\lambda(f)$ for the order and the exponent of convergence of zeros of $f(z)$ respectively. Moreover, we denote by $S(r, f)$ any real function of growth $o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure.

Many mathematicians have been interested in the value distribution of different expressions of a meromorphic function and obtained many fruitful results. In [13], Hayman discussed Picard's values of a meromorphic function and its derivatives. In particular, he proved the following result.

Theorem A. *Let $f(z)$ be a transcendental entire function. Then*

(a) *for $n \geq 3$ and $a \neq 0$, $\Psi(z) = f'(z) - a(f(z))^n$ assumes all finite values infinitely often.*
(b) *for $n \geq 2$, $\Phi(z) = f'(z)(f(z))^n$ assumes all finite values except possibly zero infinitely often.*

Recently, a number of papers (including [1–11,14,15,17,18,21,22]) have focused on complex difference equations and difference analogues of Nevanlinna theory. Bergweiler–Langley [2] first investigated the existence of zeros of $\triangle f(z)$ and $\frac{\triangle f(z)}{f(z)}$ and obtained many profound and significant results. These results may be viewed as discrete analogues of the relative existing theorem on the zeros of f'. Later on, many further results were obtained (see e.g. [3–5,17,18]). Also, Halburd–Korhonen [11] posed that the study of zeros distribution of complex difference operator plays an important role in the further study of complex differences and difference equations.

In particular, Laine–Yang [17] proved the following Theorem B, which can be considered as a difference counterpart of Theorem A(b).

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* Corresponding author.
E-mail addresses: zhengxiumin2008@sina.com (X.-M. Zheng), chzx@vip.sina.com (Z.-X. Chen).

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Theorem B. *Let $f(z)$ be a transcendental entire function of finite order, and c be a non-zero complex constant. Then for $n \geq 2$, $\Phi_1(z) = f(z+c)(f(z))^n$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.*

Later on, Chen et al. complemented the case $n = 1$ of Theorem B and obtained the following Theorem C in [3].

Theorem C. *Let $f(z)$ be a transcendental entire function of finite order, and let $c \in \mathbb{C} \setminus \{0\}$ be a complex constant. If $f(z)$ has infinitely many multi-order zeros, then $\Phi_2(z) = f(z)f(z+c)$ assumes every value $a \in \mathbb{C}$ infinitely often.*

In this paper, we continue to find a difference counterpart of Theorem A(a) and generalize it to more general cases to some extent.

Theorem 1. *Let f be a transcendental entire function of finite order, and a, c be non-zero complex constants. Then for any integer $n \geq 3$,*

$$\Psi_1(z) = f(z+c) - a(f(z))^n$$

assumes all finite values $b \in \mathbb{C}$ infinitely often.

By using the same reasoning method as Theorem 1, we immediately generalize it to the next Theorem 2.

Theorem 2. *Let f be a transcendental entire function of finite order, $a \in \mathbb{C} \setminus \{0\}$, $m, n \in \mathbb{N}^+$ and $c_1, c_2, \ldots, c_m$ be complex constants satisfying that at least one of them is non-zero. If $m < \frac{n-1}{2-\frac{1}{n}}$, then*

$$\Psi_2(z) = f(z+c_1)f(z+c_2)\cdots f(z+c_m) - a(f(z))^n$$

assumes all finite values $b \in \mathbb{C}$ infinitely often.

By using a different reasoning from Theorems 1 and 2, we obtain more general results to some extent under additional assumptions.

Theorem 3. *Let f be a transcendental entire function of finite order, $a \in \mathbb{C} \setminus \{0\}$, $m, n \in \mathbb{N}^+$ and $c_1, c_2, \ldots, c_m$ be complex constants such that at least one of them is non-zero. If $N\left(r, \frac{1}{f}\right) = S(r, f)$, $n \neq m$, then*

(a) *for $|n - m| = 1$, $\Psi_2(z)$ has infinitely many zeros;*
(b) *for $\min\{n, m\} = d \geq 2$, $\Psi_2(z)$ assumes every non-zero value $b \in \mathbb{C}$ infinitely often.*

Theorem 4. *Let f be a transcendental entire function of finite order $\sigma(f)$ with a Borel exceptional value s, $a \in \mathbb{C} \setminus \{0\}$, $c_1, c_2, \ldots, c_m$ be complex constants satisfying that at least one of them is non-zero. Then for $1 \leq m < n$ and every $b(\neq s^m - as^n) \in \mathbb{C}$, $\Psi_2(z)$ assumes the value b infinitely often and $\lambda(\Psi_2 - b) = \sigma(f)$.*

Discussion. We are settled with the case $n = m$ for Theorem 3, the case $n \leq m$ for Theorem 4 and the case of meromorphic functions for Theorems 1–4 as open questions. For example, in [4], Chen–Shon investigated the zeros distribution of $f(z+c_1)f(z+c_2) - (f(z))^2$, where f is a transcendental meromorphic function of order less than 1, without the assumption that $N\left(r, \frac{1}{f}\right) = S(r, f)$ especially.

2. Lemmas for proofs of theorems

Lemma 1 ([7]). *Let f be a meromorphic function with order $\sigma = \sigma(f) < \infty$ and let η be a fixed non-zero complex number, then for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

Remark 1. We immediately have by Lemma 1 that $\sigma(f(z+\eta)) = \sigma = \sigma(f)$, whenever f is of finite order.

Lemma 2 ([10]). *Let $T : (0, +\infty) \to (0, +\infty)$ be a non-decreasing continuous function, $s > 0$, $\alpha < 1$, and let $F \subset \mathbb{R}^+$ be the set of all r such that $T(r) \leq \alpha T(r+s)$. If the logarithmic measure of F is infinite, that is $\int_F \frac{dr}{r} = \infty$, then $\overline{\lim}_{r\to\infty} \frac{\log T(r)}{\log r} = \infty$.*

Remark 2. We immediately have by Lemma 2 that

$$T(r+s, f) = (1+o(1))T(r, f) \quad \text{and} \quad N(r+s, f) = (1+o(1))N(r, f)$$

hold for $s > 0$ and all r outside of a set with finite logarithmic measure, whenever f is of finite order.