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Inverse limits and statistical properties for chaotic implicitly defined economic models

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ABSTRACT

In this paper, we study the dynamics and ergodic theory of certain economic models which are implicitly defined. We consider 1-dimensional and 2-dimensional overlapping generations models, a cash-in-advance model, heterogeneous markets and a cobweb model with adaptive adjustment. We consider the inverse limit spaces of certain chaotic invariant fractal sets and their metric, ergodic and stability properties. The inverse limits give the set of intertemporal perfect foresight equilibria for the economic problem considered. First we show that the inverse limits of these models are stable under perturbations. We then employ utility functions on inverse limits in our case. We give two ways to rank such utility functions. First, when perturbing certain dynamical systems, we rank utility functions in terms of their *average values* with respect to invariant probability measures on inverse limits, especially with respect to measures of maximal entropy. For families of certain unimodal maps we can adjust both the discount factor and the system parameters in order to obtain maximal average value of the utility. The second way to rank utility functions (for more general maps on hyperbolic sets) will be to use equilibrium measures of these utility functions on inverse limits; they optimize average values of utility functions while *at the same time* keeping the disorder in the system as low as possible in the long run.

1. Non-invertible economic models. Outline of main results

Non-invertible dynamical systems have found many applications in various economic models, in which the equilibrium at time $t + 1$ is not uniquely defined by the one at time t; instead there may exist several such optimal states at time $t + 1$. We refer to these systems as *implicitly defined economic systems*.

In this paper, we study the dynamical and ergodic properties of such systems which present chaotic behaviour on certain invariant sets. Among the economic systems with non-invertible (or *backward*) dynamics there are the 1-dimensional and the 2-dimensional overlapping generations models, the cash-in-advance model, the cobweb model with adaptive adjustment and a class of models representing heterogeneous market agents with adaptively rational rules. The common feature of all these models is that they are given by non-invertible dynamical systems and present chaotic behaviour. In some of these models, we have *hyperbolic horseshoes* (as in the cobweb model, see [1,2]), in others *transversal homoclinic/heteroclinic orbits from saddle points* (see the heterogeneous market model, [3]), or yet in others there exist *snap-back repellers*, as in the 1-dimensional and 2-dimensional overlapping generations models for certain offer curves (see [4]). Also in the case of unimodal maps modelling some overlapping generations scenarios, we have chaotic behaviour on *repelling invariant Cantor sets* (as for the logistic map F_ν with $\nu > 4$, see [5,6]).

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For such non-invertible dynamical systems, the inverse limits are very important since they provide a natural framework in which the system "unfolds" and they give sequences of intertemporal equilibria. Also as we will see they are important since many results from the theory of expansive homeomorphisms can be applied on inverse limits, in particular those about lifts of invariant measures. *Equilibrium measures* of Holder potentials are significant examples of invariant measures and they are very important for the evolution of the system. For instance, the measure of maximal entropy gives the distribution on the phase space associated to "maximal chaos". The Sinai–Ruelle–Bowen measure (see [7,8]) on a hyperbolic attractor or of an Anosov diffeomorphism is again an equilibrium measure (for the unstable potential), and gives the limiting distribution of the forward iterates of Lebesgue-almost all points in a neighbourhood of the attractor. Thus it is a *natural measure* or *physical measure* of the system since it can be actually observed in experiments/computer simulations.

Another important feature for economic dynamical systems is that of *stability*. We are interested if a certain model is *stable* on invariant sets at small fluctuations. In our case, since we work with infinite sequences of intertemporal equilibria, one would like to have stability of the shifts on the inverse limit spaces.

The standard method of studying evolution of a system in economics is to use random (stochastic) dynamical systems which transfers exogenous random "shocks" to the system. However a system which presents chaotic behaviour, has also complicated *endogenous* fluctuations.

Also given an implicitly defined economic system with its inverse limit of intertemporal equilibria and an utility function on these equilibria, a central government/central bank may want to find a *distribution on the set of intertemporal equilibria* which maximizes the average value of the utility, but at the same time keeps the disorder in the system as little as possible in the long run. If $W(\cdot)$ is a utility function on $\hat{\Lambda}$ and $\hat{\mu}$ is a $\hat{f}$-invariant measure on $\hat{\Lambda}$ with measure-theoretic entropy $h_{\hat{\mu}}$, then the maximum in $\hat{\mu}$ of the expression

$$\int_{\hat{\Lambda}} W(\hat{x})d\hat{\mu}(\hat{x}) + h_{\hat{\mu}}$$

is attained for the *equilibrium measure* $\hat{\mu}_W$ of W (see for instance [9] for the Variational Principle for Topological Pressure). So the equilibrium measures may provide a good way to do that, and we will be able to give geometric and statistical properties of these measures. One of the defining characteristics of chaos is sensitive dependence on initial conditions, that is, even if we start with two initial states that are quite close to each other, still over time, they may become very far from each other. The equilibrium measures will permit us to estimate the *measures of sets of points which stay close* up to n iterations.

We will use the notion of *chaotic map* several times. We say that f is *chaotic* on an invariant set X if f is topologically transitive on X and f has sensitive dependence on initial conditions (see for e.g. [6]).

The *main sections and results* of the paper are the following:

First we review some important economic models with non-invertible dynamics, like the overlapping generations model, the cash-in-advance model, the cobweb model with adaptive adjustments and the heterogeneous market model. A common feature of all these models is the backward dynamics born out of implicitly defined difference equations. Also in many instances we have chaotic invariant sets for these models, given by horseshoes, or by snap-back repellers, or by transverse homoclinic orbits. Therefore we have hyperbolicity on certain invariant sets or conjugation of an iterate with the shift on some 1-sided symbol space Σ_m^+.

In Theorem 1 we will prove that by slightly *perturbing* the parameters of these difference equation, we obtain again the same dynamical properties, for instance density of periodic points, topological transitivity, etc.

We study then *utility functions on inverse limits* for non-invertible economic systems. Invariant measures for a dynamical system are very important since they preserve the ergodic and dynamical properties of the system in time; in fact from any measure one can form canonically an invariant measure by a well-known procedure (see for e.g. [9]). We will give *two options to rank utility functions*: one using average values with respect to invariant probability borelian measures, especially measures of maximal entropy (which best describe the chaotic distribution of the system over time), and another by using equilibrium measures of the utility functions, which give the best average value while keeping the system as under control as possible.

The first option is given in Theorem 2 where we rank utility functions of systems given by certain unimodal maps according to their average values with respect to invariant borelian measures $\hat{\mu}$ on the inverse limits, especially with respect to measures of maximal entropy. For certain expanding systems, namely for logistic maps $F_\nu, \ \nu > 4$ we are able to compare in Corollary 1 the *average utility values* with respect to the corresponding measures of maximal entropy when perturbing both the discount factor β of the utility W, as well as the system parameter $\nu > 4$.

Then in Theorem 3 we will prove that the inverse limits of certain invariant sets for these models are *expansive*, and have also the *specification property*. This will allow us in Theorem 4 to show that given a Holder continuous potential, we can associate to it a special probability measure called an *equilibrium measure* (see [9,10] for definitions). This equilibrium measure can be estimated precisely, on sets of points remaining close to each other up to a certain positive iterate (i.e. on Bowen balls). We can apply these results to utility functions from economics, which are shown to be Holder potentials.

The second option to rank utility functions we consider, is to maximize the *ratio* between the exponential of the average value with respect to $\hat{\mu}$ and the measure $\hat{\mu}$ of the set of points from the inverse limit that remain close up to a certain number of iterates. In this way we find the distribution $\hat{\mu}$ which maximizes the average utility value but *at the same time* keeps the "disorder" of the system (i.e. the entropy of $\hat{\mu}$) as small as possible (equivalently the measure of the set of points which shadow x up to order n, is as large as possible). Equilibrium measures of Holder potentials on the inverse limit have also