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Extrapolation and local acceleration of an iterative process for common fixed point problems

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ABSTRACT

We consider sequential iterative processes for the common fixed point problem of families of *cutter operators* on a Hilbert space. These are operators that have the property that, for any point $x \in \mathcal{H}$, the hyperplane through Tx whose normal is $x - Tx$ always "cuts" the space into two half-spaces, one of which contains the point x while the other contains the (assumed nonempty) fixed point set of T. We define and study generalized relaxations and extrapolation of cutter operators, and construct extrapolated cyclic cutter operators. In this framework we investigate the Dos Santos local acceleration method in a unified manner and adopt it to a composition of cutters. For these, we conduct a convergence analysis of successive iteration algorithms.

1. Introduction

Our point of departure that motivates us in this work is a local acceleration technique of Cimmino's [1] well-known simultaneous projection method for linear equations. This technique is referred to in the literature as the *Dos Santos* (DS) method, see [2] and [3, Section 7], although Dos Santos attributes it, in the linear case, to De Pierro's Ph.D. Thesis [4]. The method essentially uses the line through each pair of consecutive Cimmino iterates, and chooses the point on this line which is closest to the solution x^* of the linear system $Ax = b$. The nice thing about it is that existence of the solution of the linear system must be assumed, but the method does not need the solution point x^* in order to proceed with the locally accelerated DS iterative process. This approach was also used by Appleby and Smolarski [5]. On the other hand, while trying to be as close as possible to the solution point x^* in each iteration, the method is not yet known to guarantee overall acceleration of the process. Therefore, we call it a *local acceleration* technique. In all the above references the DS method works for *simultaneous projection methods*, and our first question was whether it can also work with *sequential projection methods*. Once we discovered that this is possible, the next natural question for sequential locally accelerated DS iterative process is how far the principle of the DS method can be upgraded from the linear equations model. Can it work for closed and convex sets feasibility problems? That is, can the locally accelerated DS method be preserved if orthogonal projections onto hyperplanes are replaced by metric projections onto closed and convex sets? Furthermore, can the latter be replaced by subgradient projectors onto closed and convex sets in a valid locally accelerated DS method? Finally, can the theory be extended to handle common fixed point problems? If so, for which classes of operators?

In this study, we answer these questions by focusing on the class of operators $T : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, that have the property that, for any $x \in \mathcal{H}$, the hyperplane through Tx whose normal is $x - Tx$ always "cuts" the space into two half-spaces, one of which contains the point x while the other contains the (assumed nonempty) fixed point set

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of T. This explains the name *cutter operators* or *cutters* that we introduce here. These operators, introduced and investigated by Bauschke and Combettes [6, Definition 2.2] and by Combettes [7], play an important role in optimization and feasibility theory, since many commonly used operators are actually cutters. We define generalized relaxations and extrapolation of cutter operators and construct *extrapolated cyclic cutter operators*. For these cyclic extrapolated cutters, we present convergence results of successive iteration processes for common fixed point problems.

Finally, we show that these iterative algorithmic frameworks can handle sequential locally accelerated DS iterative processes, and thus cover some of the earlier results about such methods and present some new ones.

The paper is organized as follows. In Section 2, we give the definition of cutter operators and present some of their properties that will be used here. Section 3 presents the main convergence results. Applications to specific convex feasibility problems, which show how the locally accelerated DS iterative processes follow from our general convergence results, are furnished in Section 4.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and with norm $\|\cdot\|$. Given $x, y \in \mathcal{H}$, we denote

$$H(x, y) := \{u \in \mathcal{H} \mid \langle u - y, x - y\rangle \leq 0\}\,. \tag{1}$$

Definition 1. An operator $T : \mathcal{H} \to \mathcal{H}$ is called a *cutter operator* or, in short, a *cutter*, if

$$\mathrm{Fix}T \subseteq H(x, Tx) \quad \text{for all } x \in \mathcal{H}, \tag{2}$$

where Fix$T$ is the fixed point set of T, or, equivalently,

$$q \in \mathrm{Fix}T \text{ implies that } \langle Tx - x, Tx - q\rangle \leq 0 \quad \text{for all } x \in \mathcal{H}. \tag{3}$$

The inequality in (3) can be written equivalently in the form

$$\langle Tx - x, q - x\rangle \geq \|Tx - x\|^2. \tag{4}$$

The class of cutter operators is denoted by $\mathcal{T}$, i.e.,

$$\mathcal{T} := \{T : \mathcal{H} \to \mathcal{H} \mid \mathrm{Fix}T \subseteq H(x, Tx) \text{ for all } x \in \mathcal{H}\}\,. \tag{5}$$

The class $\mathcal{T}$ of operators was introduced and investigated by Bauschke and Combettes in [6, Definition 2.2] and by Combettes in [7]. Yamada and Ogura [8] and Mainge [9] named the cutters *firmly quasi-nonexpansive* operators. These operators were named *directed operators* in Zaknoon [10] and were further employed under this name by Segal [11] and Censor and Segal [12–14]. Cegielski [15, Definition 2.1] named and studied these operators as *separating operators*. Since both *directed* and *separating* are key words of other widely used mathematical entities, we decided in [16] to use the term *cutter operators*. This name can be justified by the fact that the bounding hyperplane of $H(x, Tx)$ "cuts" the space into two half-spaces, one which contains the point x while the other contains the set Fix$T$. We recall definitions and results on cutter operators and their properties as they appear in [6, Proposition 2.4] and [7], which are also sources for further references.

Bauschke and Combettes [6] showed the following.

(i) The set of all fixed points of a cutter operator with nonempty Fix$T$ is a closed and convex subset of $\mathcal{H}$, because Fix$T = \cap_{x\in\mathcal{H}} H(x, Tx)$.

Denoting by I the identity operator,

$$\text{if } T \in \mathcal{T} \text{ then } I + \lambda(T - I) \in \mathcal{T} \quad \text{for all } \lambda \in [0, 1]. \tag{6}$$

This class of operators is fundamental, because many common types of operator arising in convex optimization belong to the class, and because it allows a complete characterization of Fejér-monotonicity [6, Proposition 2.7]. The localization of fixed points is discussed by Goebel and Reich in [17, pp. 43–44]. In particular, it is shown there that a firmly nonexpansive operator, namely, an operator $T : \mathcal{H} \to \mathcal{H}$ that fulfills

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle \quad \text{for all } x, y \in \mathcal{H}, \tag{7}$$

and has a fixed point, satisfies (3) and is, therefore, a cutter operator. The class of cutter operators includes, additionally, according to [6, Proposition 2.3], among others, the resolvents of a maximal monotone operators, the orthogonal projections, and the subgradient projectors. Another family of cutters appeared recently in [13, Definition 2.7]. Note that every cutter operator belongs to the class of operators $\mathcal{F}^0$, defined by Crombez [18, p. 161],

$$\mathcal{F}^0 := \{T : \mathcal{H} \to \mathcal{H} \mid \|Tx - q\| \leq \|x - q\| \text{ for all } q \in \mathrm{Fix}T \text{ and } x \in \mathcal{H}\}\,, \tag{8}$$

whose elements are called elsewhere quasi-nonexpansive or paracontracting operators. An example of a quasi-nonexpansive operator $T : \mathcal{H} \to \mathcal{H}$ is a nonexpansive one, i.e., an operator satisfying $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in \mathcal{H}$, with Fix$T \neq \emptyset$.