J. Math. Anal. Appl. 394 (2012) 835–840

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Cauchy–Schwarz inequality in semi-inner product C^*-modules via polar decomposition

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ARTICLE INFO

Article history:
Received 21 February 2012
Available online 7 May 2012
Submitted by J.D.M. Wright

Keywords:
Hilbert C^*-module
Operator inequality
Operator geometric mean
Positive operator
Kantorovich inequality

ABSTRACT

By virtue of the operator geometric mean and the polar decomposition, we present a new Cauchy–Schwarz inequality in the framework of semi-inner product C^*-modules over unital C^*–algebras and discuss the equality case. We also give several additive and multiplicative type reverses of it. As an application, we present a Kantorovich type inequality on a Hilbert C^*-module.

1. Introduction

The Cauchy–Schwarz inequality $|\langle x|y\rangle| \le \langle x|x\rangle^{\frac{1}{2}} \langle y|y\rangle^{\frac{1}{2}}$ in a semi-inner product space $(\mathscr{H}, \langle\cdot|\cdot\rangle)$ over the complex number field plays an important role in functional analysis. There are several generalizations and refinements of this classical inequality in various settings for different objects such as integrals and isotone functionals, see the monograph [1] and references therein. The notion of Hilbert C^*-module is a generalization of that of Hilbert space in which the inner product takes its values in a C^*-algebra instead of the complex numbers. A version of the Cauchy–Schwarz inequality in semi-inner product C^*-modules first appeared in [2] by utilizing the operator norm, afterwards in [3] using the assumption of commutativity, and in [4] using the assumption of invertibility.

On the other hand, spreading out an idea of Kantorovich inequality, Dragomir [5] proposed several additive and multiplicative type reverses of the Cauchy–Schwarz inequality in a pre-inner product space. Afterwards some reverse Cauchy–Schwarz type inequalities in other settings have been investigated: An application of the covariance-variance inequality to the Cauchy–Schwarz inequality was obtained by Fujii–Izumino–Nakamoto–Seo [6]. A refinement of the Cauchy–Schwarz inequality involving connections is investigated by Wada [7]. Niculescu [8], Joiţa [9], Moslehian–Persson [10] and Arambasić–Bakić–Moslehian [11] have investigated the Cauchy–Schwarz inequality and its various reverses in the framework of C^*-algebras and Hilbert C^*-modules. Some operator versions of the Cauchy–Schwarz inequality with simple conditions for the case of equality are presented by Fujii [12]. The authors of [4] gave some reverse Cauchy–Schwarz

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.04.083

inequalities and presented some Klamkin–Mclenaghan, Shisha–Mond, Cassels and Grüss type inequalities in Hilbert C^*-modules. Other related results may be found in [13,14].

In this paper, by virtue of the operator geometric mean and the polar decomposition, we present a new Cauchy–Schwarz inequality in the framework of semi-inner product C^*-modules over unital C^*-algebras and discuss the equality case. We also give several additive and multiplicative type reverses of it, see also [10]. As an application, we present a Kantorovich type inequality on a Hilbert C^*-module.

2. Preliminaries

Let us fix our notation and terminology. Let $\mathscr{A}$ be a unital C^*-algebra with the unit element e and the center $\mathcal{Z}(\mathscr{A})$. For $a \in \mathscr{A}$, we denote the real part of a by $\operatorname{Re} a = \frac{1}{2}(a + a^*)$. An element $a \in \mathscr{A}$ is called positive if it is selfadjoint and its spectrum is contained in $[0, \infty)$. For a positive element $a \in \mathscr{A}$, $a^{\frac{1}{2}}$ denotes the unique positive element $b \in \mathscr{A}$ such that $b^2 = a$. For $a \in \mathscr{A}$, we denote the absolute value of a by $|a| = (a^*a)^{\frac{1}{2}}$. If $a \in \mathcal{Z}(\mathscr{A})$ is positive, then $a^{\frac{1}{2}} \in \mathcal{Z}(\mathscr{A})$. If $a, b \in \mathcal{Z}(\mathscr{A})$ are positive, then ab is positive and $(ab)^{\frac{1}{2}} = a^{\frac{1}{2}} b^{\frac{1}{2}}$.

A complex linear space $\mathscr{X}$ is said to be an inner product $\mathscr{A}$-module (or a pre-Hilbert $\mathscr{A}$-module) if $\mathscr{X}$ is a right $\mathscr{A}$-module together with a C^*-valued map $(x, y) \mapsto \langle x, y\rangle : \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ such that

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$ $(x, y, x \in \mathscr{X}, \alpha, \beta \in \mathbb{C})$,
(ii) $\langle x, ya\rangle = \langle x, y\rangle a$ $(x, y \in \mathscr{X}, a \in \mathscr{A})$,
(iii) $\langle y, x\rangle = \langle x, y\rangle^*$ $(x, y \in \mathscr{X})$,
(iv) $\langle x, x\rangle \geq 0$ $(x \in \mathscr{X})$ and if $\langle x, x\rangle = 0$, then $x = 0$.

We always assume that the linear structures of $\mathscr{A}$ and $\mathscr{X}$ are compatible. Notice that (ii) and (iii) imply $\langle xa, y\rangle = a^*\langle x, y\rangle$ for all $x, y \in \mathscr{X}$, $a \in \mathscr{A}$. If $\mathscr{X}$ satisfies all conditions for an inner-product $\mathscr{A}$-module except for the second part of (iv), then we call $\mathscr{X}$ a semi-inner product $\mathscr{A}$-module.

In this case, we write $\| x \| := \sqrt{\| \langle x, x\rangle \|}$, where the latter norm denotes the C^*-norm of $\mathscr{A}$. If an inner-product $\mathscr{A}$-module $\mathscr{X}$ is complete with respect to its norm, then $\mathscr{X}$ is called a *Hilbert $\mathscr{A}$-module*. Three typical examples of Hilbert C^*-modules are as follows:

- Every Hilbert space is a Hilbert $\mathbb{C}$-module via $\langle x, y\rangle = \langle y|x\rangle$.
- Let $\mathscr{A}$ be a C^*-algebra. Then $\mathscr{A}$ is a Hilbert C^*-module over itself via $\langle x, y\rangle = x^*y$ $(x, y \in \mathscr{A})$.
- Let $\ell_2(\mathscr{A}) = \{(a_i)_{i\in\mathbb{N}} : \sum_{i=1}^{\infty} a_i^* a_i$ norm-converges in $\mathscr{A},\ a_i \in \mathscr{A}, i = 1, 2, \dots\}$. Then $\ell_2(\mathscr{A})$ is a Hilbert $\mathscr{A}$-module under the natural operations $\lambda(a_i) + \mu(b_i) = (\lambda a_i + \mu b_i)$, $(a_i)a = (a_i a)$ and $\langle (a_i), (b_i)\rangle = \sum_{i=1}^{\infty} a_i^* b_i$.

The theory of Hilbert C^*-modules is, however, not trivial, since some fundamental properties of Hilbert spaces such as adjointability of any bounded linear operator, the Pythagoras equality, triangle inequality for C^*-valued norm $|x| = \langle x, x\rangle^{1/2}$ $(x \in \mathscr{X})$ and decomposition into orthogonal complements are not true in the context of Hilbert C^*-modules in general. For more details on Hilbert C^*-modules, see [2].

3. Cauchy–Schwarz inequality and its reverses

Let a and b be positive elements of a C^*-algebra $\mathscr{A}$. Then the operator geometric mean $a\sharp b$ is defined by

$$a\sharp b = a^{\frac{1}{2}}(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})^{\frac{1}{2}} a^{\frac{1}{2}}$$

if a is invertible, see [15–17]. The operator geometric mean has the symmetric property: $a\sharp b = b\sharp a$, see [16]. If a commutes with b, then $a \sharp b = a^{\frac{1}{2}} b^{\frac{1}{2}}$. From this viewpoint, we would expect that the following Cauchy–Schwarz inequality in a semi-inner product C^*-module holds:

$$|\langle x, y\rangle| \leq \langle x, x\rangle \sharp \langle y, y\rangle \quad (x, y \in \mathscr{X}).$$

Unfortunately we have a counterexample. If $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ in the Hilbert $M_2(\mathbb{C})$-module $M_2(\mathbb{C})$, then we have $|\langle x, y\rangle| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\langle x, x\rangle \sharp \langle y, y\rangle = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore, we have $|\langle x, y\rangle| \not\leq \langle x, x\rangle \sharp \langle y, y\rangle$.

By virtue of the polar decomposition, we have the following Cauchy–Schwarz inequality in a semi-inner product C^*-module.

Theorem 3.1. *Let $\mathscr{X}$ be a semi-inner product $\mathscr{A}$-module over a unital C^*-algebra $\mathscr{A}$. Suppose that $x, y \in \mathscr{X}$ such that $\langle x, y\rangle = u|\langle x, y\rangle|$ is a polar decomposition in $\mathscr{A}$, i.e., $u \in \mathscr{A}$ is a partial isometry. Then*

$$|\langle x, y\rangle| \leq u^*\langle x, x\rangle u \sharp \langle y, y\rangle. \tag{3.1}$$

Under the assumption that $\mathscr{X}$ is an inner product $\mathscr{A}$-module and $\langle y, y\rangle$ is invertible, the equality in (3.1) holds if and only if $xu = yb$ for some $b \in \mathscr{A}$.