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Purely infinite simple reduced C^*-algebras of one-relator separated graphs

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ABSTRACT

Given a separated graph (E, C), there are two different C^*-algebras associated to it: the full graph C^*-algebra $C^*(E, C)$ and the reduced one $C^*_{\mathrm{red}}(E, C)$. For a large class of separated graphs (E, C), we prove that $C^*_{\mathrm{red}}(E, C)$ either is purely infinite simple or admits a faithful tracial state. The main tool we use to show pure infiniteness of reduced graph C^*-algebras is a generalization to the amalgamated case of a result on purely infinite simple free products due to Dykema.

1. Introduction

Separated graphs have been introduced in [1,2]. Their associated graph algebras provide generalizations of the usual graph C^*-algebras [3] and Leavitt path algebras [4,5] associated to directed graphs, although these algebras behave quite differently from the usual graph algebras because the range projections associated to different edges need not commute. One motivation for their introduction was to provide graph-algebraic models for C^*-algebraic analogues of the Leavitt algebras of [6]. These had been studied by various authors over the years, notably McClanahan (see [7–9]). As discussed below, another motivation was to obtain graph algebras whose structure of projections is as general as possible.

Given a finitely separated graph (E, C), two different graph C^*-algebras are considered in [2]: the *full* graph C^*-algebra $C^*(E, C)$ and the *reduced* graph C^*-algebra $C^*_{\mathrm{red}}(E, C)$. These two C^*-algebras agree in the classical, non-separated case, by [2, Theorem 3.8(2)], but they differ generally; see [2, Section 4]. In this paper, we obtain significant progress on some of the open problems raised in [2]. In particular, we completely solve [2, Problem 7.3] for the class of one-relator separated graphs, giving a characterization of the purely infinite simple reduced graph C^*-algebras corresponding to this class of separated graphs. As a special case, we deduce that the reduced C^*-algebras $C^*_{\mathrm{red}}(E(m, n), C(m, n))$, which are certain C^*-completions of the Leavitt algebras $L_{\mathbb{C}}(m, n)$ of type (m, n), are purely infinite simple for $m \neq n$ (Corollary 4.5).

Reduced graph C^*-algebras of separated graphs are defined as reduced amalgamated free products of certain usual graph C^*-algebras, with respect to canonical conditional expectations; see Section 2. The reader is referred to [10] and [11, 4.7, 4.8] for a quick introduction to the subject of reduced amalgamated free products of C^*-algebras. Sufficient conditions for a reduced free product of two C^*-algebras to be purely infinite simple were obtained in [12–14]. The main tool we use in the present paper to show our results on reduced graph C^*-algebras of separated graphs is a generalization to certain amalgamated free products of a result of Dykema [14] (see Theorem 3.3).

Let us recall the definition of a separated graph.

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Definition 1.1 (*[2, Definition 1.3]*)**.** A *separated graph* is a pair (E, C) where E is a graph, $C = \bigsqcup_{v\in E^0} C_v$, and C_v is a partition of $s^{-1}(v)$ (into pairwise disjoint nonempty subsets) for every vertex v. (In case v is a sink, we take C_v to be the empty family of subsets of $s^{-1}(v)$.)

If all the sets in C are finite, we say that (E, C) is a *finitely separated* graph. This necessarily holds if E is row-finite.

The set C is a *trivial separation* of E in case $C_v = \{s^{-1}(v)\}$ for each $v \in E^0 \setminus \mathrm{Sink}(E)$. In that case, (E, C) is called a *trivially separated graph* or a *non-separated graph*.

By its definition (see Section 2), the projections of $C^*(E, C)$ and $C^*_{\mathrm{red}}(E, C)$ satisfy some obvious relations, prescribed by the structure of the separated graph (E, C). These relations can be chosen arbitrarily, and this was one of the main motivations for the work in [1,2]. This can be formalized as follows. Let (E, C) be a finitely separated graph, and let $M(E, C)$ be the abelian monoid given by generators a_v, $v \in E^0$, and relations $a_v = \sum_{e\in X} a_{r(e)}$, for $X \in C_v$, $v \in E^0$. Then there is a canonical monoid homomorphism $M(E, C) \to \mathcal{V}(C^*(E, C))$, which is conjectured to be an isomorphism for all finitely separated graphs (E, C) (see Section 6 for a discussion on this problem). Here $\mathcal{V}(\mathcal{A})$ denotes the monoid of Murray–von Neumann equivalence classes of projections in $M_\infty(\mathcal{A})$, for any C^*-algebra $\mathcal{A}$.

Given a presentation $\langle \mathcal{X} \mid \mathcal{R}\rangle$ of an abelian conical monoid M, satisfying some natural conditions, it was shown in [1, Proposition 4.4] how to associate to it a separated graph (E, C) such that $M(E, C) \cong M$. We will now recall this construction for one-relator monoids. Let

$$\left\langle a_1, \dots, a_n \mid \sum_{i=1}^n r_i a_i = \sum_{i=1}^n s_i a_i \right\rangle$$

be a presentation of the one-relator abelian conical monoid M, where $a_1, \dots, a_n$ are free generators, and r_i, s_i are non-negative integers such that $r_i + s_i > 0$ for all i. Let (E, C) be the finitely separated graph constructed as follows:

(1) $E^0 := \{v, w_1, w_2, \dots, w_n\}$.
(2) v is a source, and all the w_i are sinks.
(3) For each $i \in \{1, \dots, n\}$, there are exactly $r_i + s_i$ edges with source v and range w_i.
(4) $C = C_v = \{X, Y\}$, where X contains exactly s_i edges $v \to w_i$ for each i, and Y contains exactly r_i edges $v \to v_i$ for each i. Thus, $E^1 = X \sqcup Y$.

We call a separated graph constructed in this way a *one-relator separated graph*. As a particular example, we may consider the presentation $\langle a \mid ma = na\rangle$, with $1 \le m \le n$. This gives rise to the separated graph $(E(m, n), C(m, n))$ considered in [2, Example 4.5], with two vertices v and w, and $n + m$ arrows from v to w, and with $C(m, n) = \{X, Y\}$ where $|X| = n$ and $|Y| = m$. The C^*-algebras $C^*(E(m, n), C(m, n))$ and $C^*_{\mathrm{red}}(E(m, n), C(m, n))$ are closely related to the C^*-algebras studied in [15,7–9], see [2, Sections 4 and 6].

In this paper, we show the following dichotomy for the reduced graph C^*-algebras of one-relator separated graphs:

Theorem 1.2. *Let (E, C) be the one-relator separated graph associated to the presentation*

$$\left\langle a_1, \dots, a_n \mid \sum_{i=1}^n r_i a_i = \sum_{i=1}^n s_i a_i \right\rangle.$$

Set $M = \sum_{i=1}^n r_i$ and $N = \sum_{i=1}^n s_i$, and assume that $2 \le M \le N$. Then $C^*_{\mathrm{red}}(E, C)$ either is purely infinite simple or has a faithful tracial state, and it is purely infinite simple if and only if $M < N$ and there is $i_0 \in \{1, \dots, n\}$ such that $s_{i_0} > 0$ and $r_{i_0} > 0$. Moreover, if $N + M \ge 5$ and $C^*_{\mathrm{red}}(E, C)$ is finite, then it is simple with a unique tracial state.

When $N + M \le 4$, the C^*-algebra $C^*_{\mathrm{red}}(E, C)$ is also simple, except for a few cases. We analyze the different possibilities in the final part of Section 5. The case where $M = 1$ corresponds to an ordinary graph C^*-algebra. Observe that in particular we get that the algebras $C^*_{\mathrm{red}}(E(m, n), C(m, n))$ are purely infinite simple when $1 < m < n$. It was suggested in [9, Example 4.3] that the simple C^*-algebras $U^{\mathrm{nc}}_{(m,n),\mathrm{red}}$, for $1 < m < n$, might be examples of finite but not stably finite C^*-algebras. In view of [2, Proposition 6.1], our main result shows in particular that the C^*-algebras $U^{\mathrm{nc}}_{(m,n),\mathrm{red}}$ are purely infinite if $m < n$.

It is worth to mention the appearance of some group C^*-algebras as graph C^*-algebras of separated graphs. As noted in [2], one example of this situation occurs when we consider the separated graph (E, C) with just one vertex and with the sets of the partition reduced to singletons. In this case, the full graph C^*-algebra $C^*(E, C)$ is just the full group C^*-algebra $C^*(\mathbb{F})$ of a free group $\mathbb{F}$ of rank $|E^1|$, while the reduced graph C^*-algebra is precisely the reduced group C^*-algebra $C^*_r(\mathbb{F})$. In the present investigation, we discover another situation in which the full and the reduced graph C^*-algebras correspond (through a Morita-equivalence) to the full and reduced group C^*-algebras of a group, respectively (see Lemma 5.5(2) and Proposition 5.6). The universal unital C^*-algebra generated by a partial isometry also appears as a full corner of the full C^*-algebra of a one-relator separated graph (see Lemma 5.5(1)).

We now outline the contents of the paper. After a section of preliminaries, we obtain in Section 3 the generalization to certain amalgamated free products of Dykema's result ([14, Theorem 3.1]). Section 4 contains the proof of pure infiniteness and simplicity for a class of reduced graph C^*-algebras of one-relator separated graphs (Theorem 4.3). It also contains a direct application of Dykema's Theorem to show that certain reduced free products of Cuntz algebras are purely infinite simple