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On Cantor sets and doubling measures

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ABSTRACT

For a large class of Cantor sets on the real-line, we find sufficient and necessary conditions implying that a set has positive (resp. null) measure for all doubling measures of the real-line. We also discuss same type of questions for atomic doubling measures defined on certain midpoint Cantor sets.

1. Introduction and notation

Our main goal in this paper is to study the size of Cantor sets on the real-line $\mathbb{R}$ from the point of view of doubling measures. Recall that a measure μ on a metric space X is called *doubling* if there is a constant $c < \infty$ such that

$$0 < \mu(B(x, 2r)) \leq c\mu(B(x, r)) < \infty$$

for all $x \in X$ and $r > 0$. Here $B(x, r)$ is the open ball with centre $x \in X$ and radius $r > 0$. We note that the collection of doubling measures on $\mathbb{R}$, and more generally, on any complete doubling metric space where isolated points are not dense, is rather rich. For instance, given $\varepsilon > 0$, there are doubling measures on $\mathbb{R}$ having full measure on a set of Hausdorff and packing dimension at most ε. See [1–4].

Let $\mathcal{D}(\mathbb{R})$ be the collection of all doubling measures on $\mathbb{R}$ and denote

$$\mathcal{T} = \{C \subset \mathbb{R} : \mu(C) = 0 \text{ for all } \mu \in \mathcal{D}(\mathbb{R})\},$$
$$\mathcal{F} = \{C \subset \mathbb{R} : \mu(C) > 0 \text{ for all } \mu \in \mathcal{D}(\mathbb{R})\}.$$

In the literature, the sets in $\mathcal{F}$ have been called quasisymmetrically thick [1,5], thick for doubling measures [6], and very fat [7] and those in $\mathcal{T}$ have been termed quasisymmetrically null [1,5], null for doubling measures [6], and thin [7]. We call $C \subset \mathbb{R}$ *thin* if $C \in \mathcal{T}$ and *fat* if $C \in \mathcal{F}$.

In this paper, we address the problems of finding sufficient and/or necessary conditions for a Cantor set $C \subset \mathbb{R}$ to be fat (resp. thin). These problems arise naturally from the study of compression and expansion properties of quasisymmetric maps $f : \mathbb{R} \to \mathbb{R}$; see [1, 13.20]. A related problem is to characterise those subsets $U \subset \mathbb{R}$ which carry nontrivial doubling measures [8, Open problem 1.18]; if $C \subset \mathbb{R}$ is a fat Cantor set, then it is easy to see that $U = \mathbb{R} \setminus C$ does not carry nontrivial doubling measures. For if it did, then one could extend any doubling measure μ on U to $\mathbb{R}$ by letting $\mu(C) = 0$, and this would contradict C being fat.

We begin by discussing thinness and fatness for the middle interval Cantor sets $C(\alpha_n)$ determined via sequences $(\alpha_n)_{n=1}^{\infty}$, $0 < \alpha_n < 1$, as follows. We first remove an open interval of length α_1 from the middle of $I_{1,1} = [0, 1]$ and denote the

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remaining two intervals by $I_{2,1}$ and $I_{2,2}$. At the kth step, $k \geq 2$, we have 2^{k-1} intervals $I_{k,1}, \ldots, I_{k,2^{k-1}}$ of length $\ell_k = 2^{-k+1}\prod_{n=1}^{k-1}(1-\alpha_n)$ and we remove an interval of length $\alpha_k\ell_k$ from the middle of each $I_{k,i}$. Finally, the *middle interval Cantor set* $C = C(\alpha_n)$ is defined by

$$C = \bigcap_{k\in\mathbb{N}}\bigcup_{i=1}^{2^k} I_{k,i}.$$

The theorem below follows by combining results of Wu [9, Theorem 1], Staples and Ward [5, Theorem 1.4], and Buckley et al. [7, Theorem 0.3]. For $0 < p < \infty$, we denote by ℓ^p the set of all sequences $(\alpha_n)_{n=1}^{\infty}$, $0 < \alpha_n < 1$, for which $\sum_{n=1}^{\infty} \alpha_n^p < \infty$.

Theorem 1.1. *Let $C = C(\alpha_n)$. Then*

(1) *C is thin if and only if $(\alpha_n) \notin \bigcup_{0<p<\infty} \ell^p$.*
(2) *C is fat if and only if $(\alpha_n) \in \bigcap_{0<p<\infty} \ell^p$.*

In a recent paper, Han et al. [6] generalised Theorem 1.1 for a broader collection of (still very symmetric) Cantor sets. Related results on thin and fat sets may be found in [3,5,7,9–12].

The known proofs for Theorem 1.1 and its generalisation in [6] rely heavily on the symmetries of the sets $C(\alpha_n)$. In this paper, we wish to consider analogues of Theorem 1.1 for Cantor sets with much less symmetry. To be more precise, we introduce the following notation. Suppose that for each $n \in \mathbb{N}$, we have a collection of closed intervals $\mathcal{I}_n = \{I_{n,i}\}_i$ with mutually disjoint interiors and open intervals $\mathcal{J}_n = \{J_{n,i} \subset I_{n,i}\}$ such that each $I_{n+1,i}$ is a subset of some $I_{n,j}$, $\bigcup \mathcal{I}_{n+1} = \bigcup \mathcal{I}_n \setminus \bigcup \mathcal{J}_n$ and that $\sup_j |I_{n,j}| \to 0$ as $n \to \infty$. We also assume that $\bigcup \mathcal{I}_1$ is bounded. We refer to $\{\mathcal{I}_n, \mathcal{J}_n\}_n$ as a *Cantor construction*. The resulting *Cantor set* is given by

$$C = C_{\{\mathcal{I}_n,\mathcal{J}_n\}} = \bigcap_n \bigcup_i I_{n,i}.$$

Given the collections $\mathcal{I}_n$ and $\mathcal{J}_n$ as above, we also denote $\mathcal{I} = \bigcup_n \mathcal{I}_n$ and $\mathcal{J} = \bigcup_n \mathcal{J}_n$. If there exists $0 < c < 1$ so that $cI_{n,i} \bigcap J_{n,i} \neq \emptyset$ for all $I_{n,i}$, we say that our Cantor construction (and set) is *nice*.[1]Here $cI_{n,i}$ denotes the interval concentric with $I_{n,i}$ and with length $c|I_{n,i}|$. Furthermore, given a sequence $0 < \alpha_n < 1$, we say that the Cantor set $C = C_{\{\mathcal{I}_n,\mathcal{J}_n\}_n}$ is (α_n)*-porous* if $|J_{n,i}| \geq \alpha_n|I_{n,i}|$ for all $I_{n,i} \in \mathcal{I}_n$ and (α_n)*-thick*, if $|J_{n,i}| \leq \alpha_n|I_{n,i}|$ for all $I_{n,i}$. Finally, C is called (α_n)*-regular* if $\lambda\alpha_n|I_{n,i}| \leq |J_{n,i}| \leq \Lambda\alpha_n|I_{n,i}|$ for all $I_{n,i}$ (here $0 < \lambda \leq \Lambda < \infty$ are constants that do not depend on n nor i). We underline that these definitions do not refer only to the set C but also to the construction of C via $\{\mathcal{I}_n, \mathcal{J}_n\}_n$.

Remarks 1.2. (a) Using our notation, it is possible that a Cantor set C contains isolated points as some of the intervals $I_{n,i}$ could be degenerated. We allow this for technical reasons although in most interesting cases, e.g. if C is nice, the set C is a true Cantor set in the sense that it has no isolated points.

(b) Observe that in our definitions, we do not impose any conditions on the number or relative size of the intervals $I_{n+1,j} \subset I_{n,i}$. Note also that $I_{n+1,i} \in \mathcal{I}_{n+1}$ does not have to be a component of any $I_{n,j} \setminus J_{n,j}$.

(c) We formulate our results for Cantor sets, but it is reasonable to speak about (α_n)-porosity and (α_n)-thickness for general subsets of $\mathbb{R}$ and not only for the ones obtained from Cantor constructions. Roughly speaking, $A \subset \mathbb{R}$ is (α_n)-porous if it is contained in an (α_n)-porous Cantor set and (α_n)-thick, if it contains an (α_n)-thick Cantor sets. See [3,5] for more details. In Section 4 we provide a notion of (α_n)-porosity which is useful in any metric space.

Our main result concerning doubling measures and Cantor sets is the following theorem.

Theorem 1.3. *Suppose that $C = C_{\{\mathcal{I}_n,\mathcal{J}_n\}}$ is a nice Cantor set. Then, for each $0 < p < \infty$, there is $\mu \in \mathcal{D}(\mathbb{R})$ and $0 < \lambda \leq \Lambda < \infty$ so that*

$$\lambda\left(\frac{|J_{n,i}|}{|I_{n,i}|}\right)^p \leq \frac{\mu(J_{n,i})}{\mu(I_{n,i})} \leq \Lambda\left(\frac{|J_{n,i}|}{|I_{n,i}|}\right)^p \tag{1.1}$$

for each $I_{n,i}$.

Remark 1.4. This result is interesting already for the middle interval Cantor sets $C(\alpha_n)$. After the submission of this paper, we were informed that for uniform Cantor sets, the result has been proved independently by Peng and Wen. See [13] for the precise formulation of their result.

Let us now discuss what can be said about the validity of Theorem 1.1 for the general Cantor sets $C_{\{\mathcal{I}_n,\mathcal{J}_n\}}$. Observe that Theorem 1.1 includes the following four statements:

[1] Geometrically, this only means that if the removed holes $J_{n,i}$ are small, then they cannot lie too close to the boundary of $I_{n,i}$.