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A strong approximation theorem for positively dependent Gaussian sequences and its applications[☆]

Ke-Ang Fu[a,*], Jie Li[b]

[a] School of Statistics and Mathematics, Zhejiang Gongshang University, Hangzhou 310018, China

[b] School of Mathematics and Statistics, Zhejiang University of Finance and Economics, Hangzhou 310018, China

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ABSTRACT

Under the polynomial decay rates and finite second moment, a strong approximation theorem for partial sums of linear positive quadrant dependent Gaussian sequences is derived. As applications, the law of the iterated logarithm, the Chung-type law of the iterated logarithm and the almost sure central limit theorem are obtained with minimal conditions.

1. Introduction

A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be positively associated (PA) if for every pair of subsets A and B of $\{1, 2, \ldots, n\}$,

$$\mathrm{Cov}(f(X_i; i \in A), g(X_j; j \in B)) \geq 0$$

whenever f and g are coordinate-wise nondecreasing and the covariance exists. An infinite family is PA if every finite subfamily is PA. The notion of PA was first introduced by Esary et al. [1]. Because of its wide application in multivariate statistical analysis and system reliability, it has received considerable attention in the past two decades. Under some covariance restrictions, a number of limit theorems have been obtained for PA sequences. We refer to [2] for the central limit theorem, [3] for the functional central limit theorem, [4–6] for the Berry–Esseen inequality and the moment equalities, and [7,8] for the law of the iterated logarithm and the strong invariance principle.

Later, Lehmann [9] and Newman [10], introduced another two simple and natural definitions of positive dependence, respectively.

Definition 1.1. Two random variables X and Y are said to be positively quadrant dependent (PQD), if for all $x, y \in \mathbf{R}$

$$\mathrm{P}(X > x, Y > y) \geq \mathrm{P}(X > x)\mathrm{P}(Y > y).$$

Definition 1.2. A sequence $\{X_n, n \geq 1\}$ is said to be linear positively quadrant dependent (LPQD) if for any disjoint finite subsets A, B of $\{1, 2, \ldots, n\}$ and any positive real numbers r_j, $\sum_{i \in A} r_i X_i$ and $\sum_{j \in B} r_j X_j$ are PQD.

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* Corresponding author.
E-mail address: fukeang@hotmail.com (K.-A. Fu).

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It is readily seen that compared to PA sequences, PQD and LPQD sequences define two strictly larger classes of random variables. Consequently, the study of limit theorems of such dependent forms is full of interest. For LPQD sequences, Newman [10] established the central limit theorem and Berry–Esseen theorem, Birkel [11,12] showed the strong law of large numbers and the functional central limit theorem, and Li and Wang [13] obtained the law of the iterated logarithm.

The purpose of this work is to investigate the strong approximation theorem for LPQD sequences. It is well known that for associated random variables, [8], combining the Berkes–Philipp approximation theorem with the Csörgő–Révész quantile transform methods, established a strong approximation theorem under exponential decay rate. Thus it is easy to get the similar strong approximation for an LPQD sequence under exponential decay rate by following their methods. In this paper we aim to reduce the condition "under exponential decay rate", and establish a strong approximation result for LPQD Gaussian sequences under polynomial decay rate and finite second moment.

The rest of this paper consists of three sections: Section 2 presents our main result and its applications, and Section 3 proves them, in turn, after preparing some necessary lemmas. Hereafter, let C, C_1, C_2 etc. denote positive constants whose values possibly vary from place to place and $\log x = \ln(x \vee e)$. The notation of $a_n \sim b_n$ means that $a_n/b_n \to 1$ as $n \to \infty$, $[x]$ denotes the largest integer of x and $\stackrel{\text{a.s.}}{=}$, $\stackrel{d}{=}$ denote "=" with probability one and in distribution, respectively.

2. Main result and its applications

Throughout this paper, let $\{X_n, n \geq 1\}$ be a sequence of centered LPQD Gaussian random variables unless stated otherwise. Remember that the sum of Gaussian sequences is still Gaussian. Denote $S_n = \sum_{i=1}^n X_i$ and $\sigma_n^2 = \mathsf{E}S_n^2$, $n \geq 1$. Now we are in a position to state our main result.

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of LPQD random variables and for each $n \geq 1$, X_n is distributed as $N(0, \mathsf{E}X_n^2)$. Suppose that $0 < C_1 \leq \inf_{i\geq 1} \mathsf{E}X_i^2 \leq \sup_{i\geq 1} \mathsf{E}X_i^2 \leq C_2 < \infty$ and for some $r \in (1, 3/2)$, $q(n) := \sup_{k\geq 1} \sum_{j:|j-k|\geq n} \mathsf{Cov}(X_j, X_k) = O(n^{-r})$. Then without changing its distribution, we can redefine the sequence $\{X_n, n \geq 1\}$ on a richer probability space together with a standard Wiener process $\{W(n), n \geq 0\}$ such that for any $0 < d < (r-1)/(3r-1)$*

$$|S_n - W(\sigma_n^2)| = o(\sigma_n^{1-d}) \quad \text{a.s.} \tag{2.1}$$

Remark 2.1. This strong approximation theorem extends previous results, obtained by Yu [8] for PA sequences, to a new dependent case. In particular, we reduce the exponential decay rates to polynomial decay rates, by assuming that it is a Gaussian sequence.

As applications, we obtain two theorems as follows, where Theorem 2.2 is about the LIL and Chung-type LIL and Theorem 2.3 is concerned with the ASCLT. Since the proofs of Theorems 2.2 and 2.3 are trivial, we omit them here.

Theorem 2.2. *Under the assumptions of Theorem 2.1, we have*

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2\sigma_n^2 \log\log n}} = 1 \quad \text{a.s.}$$

and

$$\liminf_{n\to\infty} \sqrt{\frac{8\log\log n}{\pi^2\sigma_n^2}} \max_{1\leq i\leq n} |S_i| = 1 \quad \text{a.s.}$$

Remark 2.2. By assuming the sequence is Gaussian, we get the LIL for LPQD sequence (even if non-stationary) under polynomial decay rates, extending the result of Li and Wang [13] to some extent.

Theorem 2.3. *Let $f(x)$ be a real valued a.s. continuous function such that $|f(x)| \leq \exp(\iota x^2)$, where $\iota < 1/2$, and let $\{X_n, n \geq 1\}$ be stationary. Then under the assumptions of Theorem 2.1, we have*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} f\left(\frac{S_k}{\sigma_k}\right) \stackrel{\text{a.s.}}{=} \int_{-\infty}^{+\infty} f(x) d\Phi(x),$$

where $\Phi(x)$ is the standard Gaussian distribution function.

3. Proof of Theorem 2.1

The goal of this section is to prove the relation (2.1) on a possibly richer probability space on which the sequence $\{X_n, n \geq 1\}$ is redefined (without changing its distribution). In order to do this, we introduce the blocks $H_1, I_1, H_2, I_2, \ldots$ of consecutive integers and we decompose the sum S_n into three terms containing the sums over the "big" blocks H_i, the sums over the "small" blocks I_i, and the remaining X_i's (whose sum is shown to be negligible). Set $\delta \in (0, \frac{r-1-3dr+d}{2d(r-1)})$, $\delta' \in$