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The asymptotic behavior of the singular values of the convolution operators with kernels whose Fourier transforms are rational functions[☆]

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ABSTRACT

We find the asymptotics of the singular values of convolution operators (with remainder term estimate) with kernels whose Fourier transforms are rational functions.

1. Introduction

We consider the integral operator

$$A : L^2(0, T) \to L^2(0, T)$$

defined by

$$Af(x) = \int_0^T k(x-y) f(y)\, dy.$$

Here k is the restriction on $[-T, T]$ of some function which is defined on $\mathbb{R}$ (it is also denoted by k). With $\widetilde{K}$, we denote the Fourier transform of the function k, i.e.

$$\widetilde{K}(x) = \int_{\mathbb{R}} e^{-ixt} k(t)\, dt.$$

In [1], Palcev studied the asymptotic behavior of the eigenvalues of the operator A in the case where

$$\widetilde{K}(x) = \frac{\prod_{i=1}^{r} (x-c_i)^{k_i}}{\prod_{i=1}^{\nu} (x-\alpha_i)^{m_i} \prod_{i=1}^{\mu} (x-\beta_i)^{n_i}} \tag{1}$$

where $\operatorname{Im}\alpha_i > 0$ $(i = 1, 2, \ldots, \nu)$, $\operatorname{Im}\beta_j < 0$ $(j = 1, 2, \ldots, \mu)$, $\alpha_i \neq \alpha_j$, $\beta_i \neq \beta_j$, $c_i \neq c_j$, $c_i \neq \alpha_j$, $c_i \neq \beta_j$, $\sum_{i=1}^{r} k_i = m$, $\sum_{i=1}^{\nu} m_i = q$, $\sum_{i=1}^{\mu} n_i = p$ and $p + q - m = r \geq 1$.

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In the case when $\widetilde{K}$ has poles only in the upper (lower) half-plane we write $p = 0$ $(q = 0)$.

In [1] it was proved that the asymptotic behavior of the eigenvalues of the operator A essentially depends on some relation between the values m, p, q, and that is manifested in first (main) term's spectral asymptotics. The method that is used in [1] is based on the reduction of the corresponding integral equation to the equivalent Riemann boundary problem.

If T is a compact operator on the complex Hilbert space then we denote by $s_n(T)$ (singular values of the operator T) the nth eigenvalues of the operator $|T| = (T^*T)^{\frac{1}{2}}$, i.e. $s_n(T) = \lambda_n(|T|)$.

For some classes of integral operators there are methods for finding the first term in the spectral asymptotics of the singular values (or eigenvalues; see [2–5]). However, finding higher terms is rather difficult, can be done only in some situations and depends essentially on the structure of the kernel of an operator.

In this paper we give simple method for finding the asymptotics of singular values of the operator A and, also, estimating the remainder term in the spectral asymptotics.

In contrast to the situation when eigenvalues are concerned, the main term in the asymptotics of $s_n(A)$ does not depend on the relation between m, p, q.

Of course, in the case when $p = 0$ or $q = 0$, our asymptotic formula for $s_n(A)$ holds, although in that case, A is the Volterra operator and has no eigenvalues.

The method used in paper [1] is not suitable for finding the asymptotic singular values of the operator A.

2. The main result

Theorem 1. *If the kernel k of the operator A has the property that the Fourier transform is of the form (1), then for every $\beta > 1$,*

$$s_n(A) = \left(\frac{T}{n\pi}\right)^r \left(1 + O\left(\frac{\ln^\beta n}{n}\right)\right), \quad n \to \infty$$

holds.

In the proof of Theorem 1 we need the following lemma.

Lemma 1. *If C and D are compact operators on complex Hilbert space such that*

$$s_n(C) = \frac{a}{n^\alpha} + O\left(\frac{1}{n^{\alpha+1}}\right), \quad n \to \infty,\ \alpha > 0, a > 0$$

and

$$s_n(D) = O\left(e^{-d\cdot n}\right), \quad n \to \infty$$

where $d > 0$ and does not depend on n, then for every $\beta > 1$,

$$s_n(C + D) = \frac{a}{n^\alpha} + O\left(\frac{\ln^\beta n}{n^{\alpha+1}}\right), \quad n \to \infty$$

holds.

Proof. From properties of singular values of sum of two operators (see [6,7]) it follows that for every $m, k, j \in \mathbb{N}$,

$$s_{(k+1)m+j}(C + D) \le s_{km+j}(C) + s_{m+1}(D) \tag{2}$$

$$s_{km+j}(C + D) \ge s_{(k+1)m+j}(C) - s_{m+1}(D). \tag{3}$$

If we put in (2)

$$m = m(n) = \left[\ln^\beta n\right] \quad (\beta > 1),$$
$$k = k(n) = \left[\frac{n}{m(n)}\right] - 1,$$
$$j = j(n) = n - m(n) \cdot (1 + k(n))$$

($[x]$ denotes the integral part of x) then we have $n = m(k + 1) + j$, $0 \le j(n) \le \left[\ln^\beta n\right]$, and we obtain

$$\frac{n^{\alpha+1}}{\ln^\beta n}\left(s_n(C + D) - \frac{a}{n^\alpha}\right) \le \frac{n^{\alpha+1}}{\ln^\beta n}\left(s_{km+j}(C) - \frac{a}{(km+j)^\alpha}\right) + a \cdot \frac{n^{\alpha+1}}{\ln^\beta n}\left(\frac{1}{(km+j)^\alpha} - \frac{1}{n^\alpha}\right) + \frac{n^{\alpha+1}}{\ln^\beta n} s_{m+1}(D)$$
$$= A_n + B_n + C_n.$$