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Self-adjoint, unitary, and normal weighted composition operators in several variables

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ABSTRACT

We study weighted composition operators on Hilbert spaces of analytic functions on the unit ball with kernels of the form $(1 - \langle z, w\rangle)^{-\gamma}$ for $\gamma > 0$. We find necessary and sufficient conditions for the adjoint of a weighted composition operator to be a weighted composition operator or the inverse of a weighted composition operator. We then obtain characterizations of self-adjoint and unitary weighted composition operators. Normality of these operators is also investigated.

1. Introduction

Let $\mathbb{B}_n$ denote the open unit ball in $\mathbb{C}^n$. For $\mathcal{H}$ a Banach space of analytic functions on $\mathbb{B}_n$ and φ an analytic self-map of $\mathbb{B}_n$, the composition operator C_φ is defined by $C_\varphi h = h \circ \varphi$ for h in $\mathcal{H}$ for which the function $h \circ \varphi$ also belongs to $\mathcal{H}$. Researchers have been interested in studying how the function theoretic behavior of φ affects the properties of C_φ on $\mathcal{H}$ and vice versa. When $\mathcal{H}$ is a classical Hardy space or a weighted Bergman space of the unit disk, it follows from Littlewood Subordination Theorem that C_φ is bounded on $\mathcal{H}$ (see, for example, [1, Section 3.1]). On the other hand, the situation becomes more complicated in higher dimensions. For $n \geq 2$, there exist unbounded composition operators on the Hardy and Bergman spaces of $\mathbb{B}_n$, even with polynomial mappings. The interested reader is referred to [1, Chapter 3] for these examples and certain necessary and sufficient conditions for the boundedness and compactness of C_φ.

Let $f : \mathbb{B}_n \to \mathbb{C}$ be an analytic function and let φ be as above. The weighted composition operator $W_{f,\varphi}$ is defined by $W_{f,\varphi}h = f \cdot (h \circ \varphi)$ for all $h \in \mathcal{H}$ for which the function $f \cdot (h \circ \varphi)$ also belongs to $\mathcal{H}$. Weighted composition operators have arisen in the work of Forelli [2] on isometries of classical Hardy spaces H^p and in Cowen's work [3,4] on commutants of analytic Toeplitz operators on the Hardy space H^2 of the unit disk. Weighted composition operators have also been used in descriptions of adjoints of composition operators (see [5] and the references therein). Boundedness and compactness of weighted composition operators on various Hilbert spaces of analytic functions have been studied by many mathematicians (see, for example, [6–9] and references therein). Recently researchers have started investigating the relations between weighted composition operators and their adjoints. Cowen and Ko [10] and Cowen et al. [11] characterize self-adjoint weighted composition operators and study their spectral properties on weighted Hardy spaces on the unit disk whose kernel functions are of the form $K_w(z) = (1 - \overline{w}z)^{-\kappa}$ for $\kappa \geq 1$. In [12], Bourdon and Narayan study normal weighted composition operators on the Hardy space H^2. They characterize unitary weighted composition operators and apply their characterization to describe all normal operators $W_{f,\varphi}$ in the case φ fixes a point in the unit disk.

The purpose of the current paper is to study self-adjoint, unitary and normal weighted composition operators on a class of Hilbert spaces $\mathcal{H}$ of analytic functions on the unit ball. We characterize $W_{f,\varphi}$ whose adjoint is a weighted composition

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operator or the inverse of a weighted composition operator. As a consequence, we generalize certain results in [12,10,11] to higher dimensions and also obtain results that have not been previously known in one dimension.

For any real number $\gamma > 0$, let H_γ denote the Hilbert space of analytic functions on $\mathbb{B}_n$ with reproducing kernel functions

$$K_z^\gamma(w) = K^\gamma(w, z) = \frac{1}{(1 - \langle w, z\rangle)^\gamma} \quad \text{for } z, w \in \mathbb{B}_n.$$

By definition, H_γ is the completion of the linear span of $\{K_z^\gamma : z \in \mathbb{B}_n\}$ with the inner product $\langle K_z^\gamma, K_w^\gamma\rangle = K^\gamma(w, z)$ (this is indeed an inner product due to the positive definiteness of $K^\gamma(w, z)$). It is well known that any function $f \in H_\gamma$ is analytic on $\mathbb{B}_n$ and for $z \in \mathbb{B}_n$, we have $f(z) = \langle f, K_z^\gamma\rangle$.

For any multi-index $m = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$ (here $\mathbb{N}_0$ denotes the set of non-negative integers) and $z = (z_1, \ldots, z_n) \in \mathbb{B}_n$, we write $z^m = z_1^{m_1} \cdots z_n^{m_n}$. It turns out that H_γ has an orthonormal basis consisting of constant multiples of the monomials z^m, for $m \in \mathbb{N}_0^n$. The spaces H_γ belong to the class of weighted Hardy spaces introduced by Cowen and MacCluer in [1, Section 2.1]. They are called (generalized) weighted Bergman spaces by Zhao and Zhu in [13] because of their similarities with other standard weighted Bergman spaces on the unit ball. In fact, for $\gamma > n$, H_γ is the weighted Bergman space $A^2_{\gamma-n-1}(\mathbb{B}_n)$, which consists of all analytic functions that are square integrable with respect to the weighted Lebesgue measure $(1 - |z|^2)^{\gamma-n-1}dV(z)$, where dV is the Lebesgue volume measure on $\mathbb{B}_n$. If $\gamma = n$, H_n is the usual Hardy space on $\mathbb{B}_n$. When $n \geq 2$ and $\gamma = 1$, H_1 is the so-called Drury–Arveson space, which has been given a lot of attention lately in the study of multi-variable operator theory and interpolation (see [14,15] and the references therein). For arbitrary $\gamma > 0$, H_γ coincides with the space $A^2_{\gamma-n-1}(\mathbb{B}_n)$ in [13] (we warn the reader that when $\gamma < n$, the space $A^2_{\gamma-n-1}(\mathbb{B}_n)$ is not defined as the space of analytic functions that are square integrable with respect to $(1 - |z|^2)^{\gamma-n-1}dV(z)$, since the latter contains only the zero function).

2. Bounded weighted composition operators

As we mentioned in the Introduction, the composition operator C_φ is not always bounded on H_γ of the unit ball $\mathbb{B}_n$ when $n \geq 2$. On the other hand, if φ is a linear fractional self-map of the unit ball, then it was shown by Cowen and MacCluer [16] that C_φ is bounded on the Hardy space and all weighted Bergman spaces of $\mathbb{B}_n$. It turns out, as we will show below, that for such φ, C_φ is always bounded on H_γ for any $\gamma > 0$. We will need the following characterization of H_γ, which follows from [13, Theorem 13].

For any multi-index $m = (m_1, \ldots, m_n)$ of non-negative integers and any analytic function h on $\mathbb{B}_n$, we write $\partial^m h = \frac{\partial^{|m|}h}{\partial z_1^{m_1}\cdots\partial z_n^{m_n}}$, where $|m| = m_1 + \cdots + m_n$. For any real number α, put $d\mu_\alpha(z) = (1 - |z|^2)^{-n-1+\alpha}dV(z)$, where dV is the usual Lebesgue measure on the unit ball $\mathbb{B}_n$.

Theorem 2.1. *Let $\gamma > 0$. The following conditions are equivalent for an analytic function h on $\mathbb{B}_n$.*

(a) *h belongs to H_γ.*
(b) *For some non-negative integer k with $2k + \gamma > n$, all the functions $\partial^m h$, where $|m| = k$, belong to $L^2(\mathbb{B}_n, d\mu_{\gamma+2k})$.*
(c) *For every non-negative integer k with $2k + \gamma > n$, all the functions $\partial^m h$, where $|m| = k$, belong to $L^2(\mathbb{B}_n, d\mu_{\gamma+2k})$.*

Remark 2.2. Theorem 2.1 in particular shows that for any given positive number s, the function h belongs to H_γ if and only if for any multi-index l with $|l| = s$, $\partial^l h$ belongs to $H_{\gamma+2s}$. As a consequence, $H_{\gamma_1} \subset H_{\gamma_2}$ whenever $\gamma_1 \leq \gamma_2$.

Recall that the multiplier space $\text{Mult}(H_\gamma)$ of H_γ is the space of all analytic functions f on $\mathbb{B}_n$ for which fh belongs to H_γ whenever h belongs to H_γ. Since norm convergence in H_γ implies point-wise convergence on $\mathbb{B}_n$, it follows from the closed graph theorem that f is a multiplier if and only if the multiplication operator M_f is bounded on H_γ. It is well known that $\text{Mult}(H_\gamma)$ is contained in H^∞, the space of bounded analytic functions on $\mathbb{B}_n$. For $\gamma \geq n$, it holds that $\text{Mult}(H_\gamma) = H^\infty$. This follows from the fact that for such γ the norm on H_γ comes from an integral. On the other hand, when $n \geq 2$ and $\gamma = 1$ (hence H_γ is the Drury–Arveson space), $\text{Mult}(H_\gamma)$ is strictly smaller than H^∞ (see [14, Remark 8.9] or [15, Theorem 3.3]). However we will show that if f and all of its partial derivatives are bounded on $\mathbb{B}_n$, then f is a multiplier of H_γ for all $\gamma > 0$.

Lemma 2.3. *Let f be a bounded analytic function such that for each multi-index m, the function $\partial^m f$ is bounded on $\mathbb{B}_n$. Then f belongs to* $\text{Mult}(H_\gamma)$*, and hence the operator M_f is bounded on H_γ for any $\gamma > 0$.*

Proof. Let $\gamma > 0$ be given. Choose a positive integer k such that $\gamma + 2k > n$. Let h belong to H_γ. For any multi-index m with $|m| = k$, the derivative $\partial^m(fh)$ is a linear combination of products of the form $(\partial^t f)(\partial^s h)$ for multi-indexes s, t with $s + t = m$. For such s and t, $\partial^s h$ belongs to $H_{\gamma+2|s|} \subset H_{\gamma+2k}$ (by Remark 2.2) and $\partial^t f$, which is bounded by the hypothesis, is a multiplier of $H_{\gamma+2k}$ (since $\text{Mult}(H_{\gamma+2k}) = H^\infty$). Thus, $(\partial^t f)(\partial^s h)$ belongs to $H_{\gamma+2k}$. Therefore, $\partial^m(fh)$ belongs to $H_{\gamma+2k}$. By Theorem 2.1, fh is in H_γ. Since h was arbitrary in H_γ, we conclude that f is a multiplier of H_γ. □