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On higher order fully periodic boundary value problems

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ABSTRACT

In this paper we present sufficient conditions for the existence of periodic solutions of the higher order fully differential equation

$$u^{(n)}(x) = f(x, u(x), u'(x), \ldots, u^{(n-1)}(x)),$$

with $n \geq 3$, $x \in [a, b]$ and $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ a continuous function verifying a Nagumo-type growth condition.

A new type of lower and upper solutions, eventually non-ordered, allows us to obtain, not only the existence, but also some qualitative properties on the solution. The last section contains two examples to stress the application to both cases of n odd and n even.

1. Introduction

In this work we consider the higher order periodic boundary value problem composed by the fully differential equation

$$u^{(n)}(x) = f\left(x, u(x), u'(x), \ldots, u^{(n-1)}(x)\right) \tag{1}$$

for $n \geq 3$, $x \in I := [a, b]$, and $f : I \times \mathbb{R}^n \to \mathbb{R}$ a continuous function and the periodic boundary conditions

$$u^{(i)}(a) = u^{(i)}(b), \quad i = 0, 1, \ldots, n-1. \tag{2}$$

Higher order periodic boundary value problems have been studied by several authors in the last decades, using different types of arguments and techniques, as it can be seen in [1–3] for variational methods, in [4–17], for first and higher order equations and in [18–20] for a linear or quasi-linear nth order periodic problem. A fully nonlinear differential equation of higher order as in (1) was studied in some works, such as, for instance, [21], for f a bounded and periodic function verifying different assumptions for n even or odd. Moreover, in [22], the nonlinear part f of (1) must verify the following assumptions:

(A_1) There are continuous functions $e(x)$ and $g_i(x, y)$, $i = 0, \ldots, n-1$, such that

$$|f(x, y_0, \ldots, y_{n-1})| \leq e(x) + \sum_{i=0}^{n-1} g_i(x, y_i)$$

with

$$\lim_{|y| \to \infty} \sup_{x \in [0,1]} \frac{|g_i(x, y)|}{|y|} = r_i \geq 0, \quad i = 0, 1, \ldots, n-1.$$

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(A_2) There is a constant $M > 0$ such that, for $x \in [0, 1]$,

$$f(x, y_0, 0, \ldots, 0) > 0, \quad \text{for } y_0 > M,$$

and

$$f(x, y_0, 0, \ldots, 0) < 0, \quad \text{for } y_0 < -M.$$

(A_3) There are real numbers $L \geq 0$, $\alpha > 0$ and $a_i \geq 0$, $i = 1, \ldots, n-1$, such that

$$|f(x, y_0, \ldots, y_{n-1})| \geq \alpha |y_0| - \sum_{i=1}^{n-1} a_i |y_i| - L,$$

for every $x \in [0, 1]$ and $(y_0, \ldots, y_{n-1}) \in \mathbb{R}^n$.

The arguments followed in this paper allow more general nonlinearities, namely, f does not need to have a sublinear growth in $y_0, \ldots, y_{n-1}$ (as in (A_1)) or change sign (as in (A_2)). In fact, condition (10) in our main result (see Theorem 4) refers an, eventually, opposite monotony to (A_2) and improves the existent results in the literature for periodic higher order boundary value problems. In short, our technique is based on lower and upper solutions not necessarily ordered, in the topological degree theory, like it was suggested, for example, in [23,24], and has the following key points:

- A Nagumo-type condition on the nonlinearity, useful to obtain an *a priori* estimation for the $(n-1)$th derivative and to define an open and bounded set where the topological degree is well defined.
- A new kind of definition of lower and upper solutions, required to deal with the absence of a definite order for lower and upper functions and their derivatives up to the $(n-3)$th order. We remark that with such functions it is only required boundary data for the derivatives of order $n-2$ and $n-1$. Therefore the set of admissible functions for lower and upper solutions is more general.
- An adequate auxiliary and perturbed problem, where the truncations and the homotopy are extended to some mixed boundary conditions, allowing an invertible linear operator and the evaluation of the Leray–Schauder degree.

The last section contains two examples to emphasize that these results cover some cases in the literature where it is needed to particularize if n is odd and/or even.

2. Definitions and *a priori* bounds

It is introduced in this section, a Nagumo-type growth condition, initially presented in [25], and now useful to obtain an *a priori* estimate for the $(n-1)$th derivative.

Definition 1. A continuous function $f : I \times \mathbb{R}^n \to \mathbb{R}$ is said to satisfy the Nagumo type condition in

$$E = \left\{(x, y_0, \ldots, y_{n-1}) \in I \times \mathbb{R}^n : \gamma_i(x) \leq y_i \leq \Gamma_i(x),\ i = 0, 1, \ldots, n-2\right\}, \tag{3}$$

with $\gamma_i(x)$ and $\Gamma_i(x)$ continuous functions such that,

$$\gamma_i(x) \leq \Gamma_i(x), \quad \text{for } i = 0, 1, \ldots, n-2 \text{ and every } x \in I, \tag{4}$$

if there exists a real continuous function $h_E : [0, +\infty[\to]0, +\infty[$ such that

$$|f(x, y_0, \ldots, y_{n-1})| \leq h_E(|y_{n-1}|), \quad \text{for every } (x, y_0, \ldots, y_{n-1}) \in E, \tag{5}$$

with

$$\int_0^{+\infty} \frac{s}{h_E(s)} ds = +\infty. \tag{6}$$

In the following we denote

$$\|w\|_\infty := \sup_{x \in I} |w(x)|.$$

The *a priori* bound is given by the next lemma:

Lemma 2 ([24, Lemma 1]). *Consider $\gamma_i, \Gamma_i \in \mathcal{C}(I, \mathbb{R})$, for $i = 0, 1, \ldots, n-1$, such that (4) holds and E is defined by (3). Assume that, for some $k > 0$, there is $h_E \in \mathcal{C}([0, +\infty[, [k, +\infty[)$, such that*

$$\int_\eta^{+\infty} \frac{s}{h_E(s)} ds > \max_{x \in I} \Gamma_{n-2}(x) - \min_{x \in I} \gamma_{n-2}(x),$$