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Existence of weak solutions for a quasilinear equation in $\mathbb{R}^N$

M. Massar [a,*], N. Tsouli [a], A. Hamydy [b]

[a] University Mohamed I, Faculty of Sciences, Department of Mathematics, Oujda, Morocco
[b] CfI, Tetouane, Morocco

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ABSTRACT

This paper studies the p-Laplacian equation

$$-\Delta_p u + \lambda V_\lambda(x)\,|u|^{p-2}\,u = f(x,u) \text{ in } \mathbb{R}^N,$$

where $1 < p < N$, $\lambda \geq 1$ and $V_\lambda(x)$ is a nonnegative continuous function. Under some conditions on $f(x,u)$ and $V_\lambda(x)$, we prove the existence of nontrivial solutions for λ sufficiently large.

1. Introduction and main results

In this paper, we consider the following p-Laplacian equation

$$-\Delta_p u + \lambda V_\lambda(x)|u|^{p-2}u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $1 < p < N,\ \lambda \geq 1,\ V_\lambda \in C(\mathbb{R}^N, \mathbb{R})$ and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$.

We assume that the potential $V_\lambda(x)$ and $f(x,u)$ satisfy the following conditions.

$(\mathcal{V}_1)$ $0 \leq V_\lambda(x)$ for all $x \in \mathbb{R}^N$ and $\lambda \geq 1$.

$(\mathcal{V}_2)$ There exists $M > 0$ such that for all $\lambda \geq 1,\ |\Omega_{M,\lambda}| < \infty$, where

$$\Omega_{M,\lambda} = \left\{x \in \mathbb{R}^N / V_\lambda(x) \leq M\right\}.$$

$(\mathcal{V}_3)$ $\lim_{\lambda\to\infty} V_\lambda(0) = 0$.

There exist a positive function $m(x) \in L^\infty_{loc}(\mathbb{R}^N)$ and constants $C_0,\ R_0 > 0,\ \alpha > 1$ such that

$(\mathcal{V}_4)$ $m(x) \leq C_0\left(1 + (V_\lambda(x))^{1/\alpha}\right)$ for all $|x| \geq R_0$ and $\lambda \geq 1$.

(f$_1$) There exists $q \in (p, p^\sharp)$, with $p^\sharp := p^\star - \frac{p^2}{\alpha(N-p)}$ and $p^\star := \frac{Np}{N-p}$, such that
$$|f(x,t)| \leq C_0 m(x)(1 + |t|^{q-1}) \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R}.$$

(f$_2$) $\frac{f(x,t)}{m(x)} = o(|t|^{p-1})$ as $t \to 0$ uniformly in x.

(f$_3$) There exist $\mu_0, \mu > p$ and a positive continuous function $\gamma_0(x)$ such that
$$F(x,t) \geq \gamma_0(x)|t|^{\mu_0} \quad \text{and} \quad \mu F(x,t) \leq tf(x,t) \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R},$$
where $F(x,t) = \int_0^t f(x,s)ds$.

* Corresponding author.
E-mail addresses: massarmed@hotmail.com (M. Massar), tsouli@hotmail.com (N. Tsouli), a.hamydy@yahoo.fr (A. Hamydy).

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An example of functions satisfying the assumptions (f_1)–(f_3) is given by

$$f(x,u) = m_0(x)|u|^{s-2}u,$$

where $m_0(x)$ is a positive continuous function and $p < s < p^\sharp$.

Set

$$\mathcal{F} = \left\{\delta > p/\text{there exists a positive continuous function } \gamma(x) \text{ such that } F(x,t) \geq \gamma(x)|t|^\delta \text{ for } x \in \mathbb{R}^N,\ t \in \mathbb{R}\right\}.$$

By (f_1) and (f_3), we see that $\mu_0 \in \mathcal{F}$ and $\delta \leq q$ for all $\delta \in \mathcal{F}$.

The investigation of equations of the form (1.1) has been motivated by searching wave solutions for the nonlinear Schrödinger equations; see [1–3]. Many works have been devoted to the case $p = 2$; see [4–9]. The quasilinear case $p \in (1, N)$ appears in a variety of applications, such as non-Newtonian fluids, image processing, nonlinear elasticity and reaction–diffusion; see [10] for more details. In the paper [11], Liu consider the p-Laplacian equation $(1 < p < N)$

$$-\Delta_p u + V(x)|u|^{p-2}u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.2}$$

with a potential which is periodic or has a bounded potential well. Without assuming the Ambrosetti–Rabinowitz type condition and the monotonicity of the function $t \to \frac{f(x,t)}{|t|^{p-1}}$, the author proved the existence of ground states of (1.2). Another p-Laplacian equation with potential was considered by Wu and Yang [12]

$$-\Delta_p u + \lambda V(x)|u|^{p-2}u = |u|^{q-2}u, \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $2 \leq p < q < p^\star$ and the potential $V(x)$ is bounded. Using a concentration-compactness principle from critical point theory, they proved existence, multiplicity and concentration of solutions of (1.3). For more results we refer the reader to [13,14,12,15] and references therein. In the present paper, we are going to study the existence of nontrivial solutions of (1.1). The results of this paper may be considered as generalization of the results obtained by Sirakov [8]. Here we consider the situation when the potential is sufficiently large at infinity. Our method is mainly based on variational arguments.

The main results of this paper are the following theorems.

Theorem 1.1. *Assume that (f_1)–(f_3), $(\mathcal{V}_1)$–$(\mathcal{V}_4)$ hold, and $q \in \mathcal{F}$. Then there exists $\lambda_0 \geq 1$, depending only on the various constants involved in the assumptions, such that (1.1) has a nontrivial solution, for any $\lambda \geq \lambda_0$.*

In the next theorem we will remove the hypothesis $q \in \mathcal{F}$ and strengthen $(\mathcal{V}_3)$ by replacing it with a more precise condition about the behaviour of $V_\lambda(x)$ near the origin, for λ sufficiently large.

Theorem 1.2. *Assume that (f_1)–(f_3), $(\mathcal{V}_1)$, $(\mathcal{V}_2)$, $(\mathcal{V}_4)$ hold, and $(\mathcal{V}_5)$ there exist constants $C_1, \eta_0, \kappa > 0$ such that*

$$V_\lambda(x) \leq C_1\left(|x|^\kappa + \lambda^{-\frac{\kappa}{\kappa+p}}\right) \quad \text{for all } |x| \leq \eta_0 \lambda^{-\frac{1}{\kappa+p}},$$

and

$$\frac{p}{\kappa+p}\left(\frac{\delta_0}{\delta_0 - p} - \frac{N}{p}\right) < \frac{q}{q-p} - \frac{N}{p},$$

for some $\delta_0 \in \mathcal{F}$.

Then there exists $\lambda_0 \geq 1$, depending only on the various constants involved in the assumptions, such that (1.1) has a nontrivial solution, for any $\lambda \geq \lambda_0$.

2. Preliminary results

We look for solutions of (1.1) in the following subspace

$$X_\lambda = \left\{u \in W^{1,p}(\mathbb{R}^N) \Big/ \int_{\mathbb{R}^N} V_\lambda(x)|u|^p dx < \infty\right\},$$

endowed with the norm

$$\|u\|_\lambda = \left(\int_{\mathbb{R}^N} |\nabla u|^p + \lambda V_\lambda(x)|u|^p dx\right)^{1/p}.$$

Remark 2.1. It follows from $(\mathcal{V}_1)$, $(\mathcal{V}_2)$ and Poincaré's inequality for the set $\Omega_{M,\lambda}$ that there exists $C_\lambda > 0$ such that

$$\|u\|_{1,p} \leq C_\lambda \|u\|_\lambda \quad \text{for all } u \in X_\lambda,$$

where $\|.\|_{1,p}$ is the standard norm on $W^{1,p}(\mathbb{R}^N)$. Then the space $(X_\lambda, \|.\|_\lambda)$ is continuously embedded into $(W^{1,p}(\mathbb{R}^N), \|.\|_{1,p})$. Moreover, $(X_\lambda, \|.\|_\lambda)$ is a reflexive Banach space.